Weighted packing of the infinite grid $\mathbb{Z}^2$

S. Gravier*, E. Vandomme†

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Abstract

We introduce the concept of constant 2-labelling of a weighted graph and show how it can be used to obtain periodic sphere packing. Roughly speaking, a constant 2-labelling of a weighted graph is a 2-coloring $\{\bullet, \circ\}$ of its vertex set which preserves the sum of the weight of black vertices under some automorphisms. In this manuscript, we study this problem on weighted complete graphs and on weighted cycles. Our results on cycles allow us to determine (r, a, b)-codes in $\mathbb{Z}^2$ whenever $|a - b| > 4$ and $r \geq 2$.

Introduction

The motivation about introducing constant 2-labellings comes from covering problems in graphs. These lattest are coverings with balls of constant radius satisfying special multiplicity condition. Let $G = (V, E)$ be a graph and r, a, b be positive integers. A set $S \subseteq V$ of vertices is an (r, a, b)-*code* if every element of S belongs to exactly a balls of radius r centered at elements of S and every element of $V \setminus S$ belongs to exactly b balls of radius r centered at elements of S. Such codes are also known as (r, a, b)-covering codes [2], (r, a, b)-isotropic colorings [2] or as perfect colorings [10].

The notion of (r, a, b)-codes generalizes the notion of domination and perfect codes in graphs. An r-perfect code in a graph is nothing less than an $(r, 1, 1)$-code. Perfect codes were introduced in terms of graphs by Biggs in [3]. It was shown by Kratochvil [9] that the problem of finding an r-perfect code in graphs (*i.e.*, an $(r, 1, 1)$-code) is NP-complete. Moreover, this problem is even NP-complete in the case of bipartite graphs with maximum degree three. For more information about perfect codes, see [4, Chapter 11].

Cohen *et al.* [5] introduced a generalization of covering codes using weights, called weighted coverings (see also [4, Chapter 13]). A $(1, a, b)$-code is exactly a perfect weighted covering of radius one with weight $\left(\frac{b-a+1}{b}, \frac{1}{b}\right)$. This particular case has been much studied, for example see [6, 8] for $(1, a, b)$-codes in the multidimensional grid graphs. Moreover,

*CNRS, Institut Fourier, Grenoble, France
†University of Liège, Liège, Belgium, and Institut Fourier, Grenoble, France

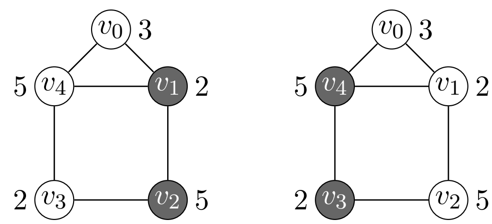


Figure 1: A coloring of a graph G and its composition with the automorphism σ.

$(1, a, b)$-codes are equivalent to $[a-1, b]$-domination sets defined by Telle. In [11], Telle proved that the following decidability problem was NP-complete : "Is it possible to decide whether a graph has an $[a, b]$-dominating set ?". It is even NP-complete when restricted to planar bipartite graphs of maximum degree three.

In this paper, we will focus on the graph of the infinite grid $\mathbb{Z}^2$. For $r \geq 2$, Puzynina [10] showed that every (r, a, b)-codes of $\mathbb{Z}^2$ are periodic (see Subsection 2.2 for a formal definition). Moreover Axenovich [2] gave a characterization all (r, a, b)-codes in $\mathbb{Z}^2$ with $r \geq 2$ and $|a - b| > 4$. We introduce the concept of constant 2-labellings to obtain a new characterization of these (r, a, b)-codes and we give all the possible values of constants a and b.

Given a graph $G = (V, E)$, a vertex v of G, a map $w : V \to \mathbb{R}$ and a subset A of the set $Aut(G)$ of all automorphisms of G, a *constant 2-labelling* of G is a mapping $\varphi : V \to \{\bullet, \circ\}$ such that

$$\sum_{\{u \in V \mid \varphi \circ \xi(u) = \bullet\}} w(u) = \sum_{\{u \in V \mid \varphi \circ \xi'(u) = \bullet\}} w(u) \quad \forall \xi, \xi' \in A_\bullet \text{ (resp. } A_\circ)$$

where $A_\bullet = \{\xi \in A \mid \varphi \circ \xi(v) = \bullet\}$ (resp. $A_\circ = \{\xi \in A \mid \varphi \circ \xi(v) = \circ\}$).

For example, let $G = (V, E)$ be a graph with $V = \{v_0, \dots, v_4\}$ represented at Figure 1. Take $v = v_0, A = Aut(G), w : V \to \mathbb{R}$ and $\varphi : V \to \{\bullet, \circ\}$ defined by $w(v_0) = 3, w(v_1) = w(v_3) = 2, w(v_2) = w(v_4) = 5$ and $\varphi(v_0) = \varphi(v_3) = \varphi(v_4) = \circ, \varphi(v_1) = \varphi(v_2) = \bullet$. It is clear that φ is a constant 2-labelling since A contains only two automorphisms, id and

$$\sigma : v_0 \mapsto v_0;\ v_1 \mapsto v_4;\ v_2 \mapsto v_3;\ v_3 \mapsto v_2;\ v_4 \mapsto v_1.$$

Constant 2-labellings are linked with distinguished colorings. A coloring is *distinguished* if it is not preserved by any non trivial automorphism of G. Introduced by Albertson and al [1], the *distinguishing number* of a graph is the smallest integer k such that there exist a distinguished coloring using k colors. This notion has already been studied in [7]. For a graph G, let φ be a non distinguished coloring of G. Then there exists a non trivial automorphism that preserves φ. If A denotes the set of automorphisms that preserve φ, then the coloring φ

is a constant 2-labelling of G. It would be interesting to consider a generalization of constant 2-labellings into constant k-labellings and their links with distinguishing numbers.

We can make some other straightforward observations about constant 2-labellings. Let a and b denote the following constants of a constant 2-labelling φ :

$$a := \sum_{\{u\in V \mid \varphi\circ\xi(u)=\bullet\}} w(u) \text{ and } b := \sum_{\{u\in V \mid \varphi\circ\xi'(u)=\bullet\}} w(u) \text{ for } \xi \in A_\bullet, \xi' \in A_\circ.$$

Recall that a coloring φ of the vertex set V is *monochromatic* if all vertices have the same color. We say that φ is *monochromatic black* (resp. *monochromatic white*) if $\varphi(v_1) = \bullet$ (resp. $\varphi(v_1) = \circ$) for all $v_1 \in V$.

Proposition 1. *Let $G = (V, E)$ be a weighted graph, $v \in V$, $w : V \to \mathbb{R}$ be the weight map and $A \subseteq Aut(G)$. If φ is a monochromatic coloring of V, then φ is a constant 2-labelling.*

In this case, the constant 2-labelling is said *trivial* and the corresponding constants are such that

- $a = \sum_{u\in V} w(u)$ and b is not defined if φ is monochromatic black,
- a is not defined and $b = 0$ if φ is monochromatic white.

The following proposition allows us to consider either a coloring φ or the coloring obtained by switching the colors of φ. Let $\sigma : \{\bullet, \circ\} \to \{\bullet, \circ\}$ be a map such that $\sigma(\bullet) = \circ$ and $\sigma(\circ) = \bullet$. The *complementary* coloring of φ is the map $\sigma \circ \varphi$ and is denoted by $\overline{\varphi}$.

Proposition 2 (Complementary property)**.** *Let $G = (V, E)$ be a weighted graph, $w : V \to \mathbb{R}$ be the weight map, $v \in V$ and $A \subseteq Aut(G)$. Set $\omega := \sum_{u\in V} w(u)$. A coloring φ is a constant 2-labelling of G with respective constants a and b if and only if the coloring $\overline{\varphi}$ is a constant 2-labelling with respective constants $\omega - b$ and $\omega - a$.*

An intersting example is the complete graph K_n.

Proposition 3. *Let $w : V(K_n) \to \mathbb{R}$, $v \in V(K_n)$ and $A = Aut(K_n)$. There is a non trivial constant 2-labelling of K_n if and only if $w(v_1) = w(v_2)$ for all $v_1, v_2 \in V \setminus \{v\}$.*

Proof. On one hand, if disctint vertices of $V \setminus \{v\}$ have distinct weights, then there doesn't exist a non trivial constant 2-labelling using exactly 2 colors. Indeed, assume that $v_1, v_2 \in V \setminus \{v\}$ have different weights and $\varphi : V \to \{\bullet, \circ\}$ is a non monochromatic coloring. Even if it means taking a composition with an automorphism, we may assume that v_1 and v_2 are of different colors such that $\bullet = \varphi(v_1) \neq \varphi(v_2) = \circ$. Let σ denote the automorphism that swaps only v_1 and v_2. We obtain that $\varphi(v) = \varphi \circ \sigma(v)$ and

$$\begin{aligned}\sum_{u\in\varphi^{-1}(\bullet)} w(u) &= \sum_{u\in\varphi^{-1}(\bullet)\setminus\{v_1,v_2\}} w(u) + w(v_1)\\ &\neq \sum_{u\in\varphi^{-1}(\bullet)\setminus\{v_1,v_2\}} w(u) + w(v_2) = \sum_{u\in(\varphi\circ\sigma)^{-1}(\bullet)} w(u).\end{aligned}$$

Therefore, φ is not a constant 2-labelling.

On the other hand, if all vertices of $V \setminus \{v\}$ have same weight, say ω, then any colouring $\varphi : V \to \{\bullet, \circ\}$ of K_n is a constant 2-labelling. Let n be the number of black vertices. It is clear that

$$\sum_{u \in \xi^{-1}(\bullet)} w(u) = w(v) + (n-1)\omega \quad \text{and} \quad \sum_{u \in \xi'^{-1}(\bullet)} w(u) = n\omega$$

for all $\xi \in A_\bullet, \xi' \in A_\circ$. $\square$

In this paper, we look at weighted cycles with p vertices denoted by $\mathcal{C}_p$. These vertices $0, \ldots, p-1$ have respectively weights $w(0), \ldots, w(p-1)$. We will represent such a cycle by the word $w(0) \ldots w(p-1)$. Let $\mathcal{R}_k$ denote a k-rotation of $\mathcal{C}_p$, i.e.,

$$\mathcal{R}_k : \{0, \ldots, p-1\} \to \{0, \ldots, p-1\} : i \mapsto i + k \bmod p.$$

If $k = 1$, we simply call a k-rotation a rotation.

In the sequel, we always take $A = \{\mathcal{R}_k \mid k \in \mathbb{Z}\}$ and $v = 0$. A coloring $\varphi : \{0, \ldots, p-1\} \to \{\bullet, \circ\}$ of a cycle $\mathcal{C}_p$ is a constant 2-labelling if, for every k-rotation of the coloring, the weighted sum of black vertices is a constant a (resp. b) whenever the vertex 0 is black (resp. white). We are interested in the following problem. Given a cycle $\mathcal{C}_p$ of p weighted vertices, can we find a non trivial constant 2-labelling ?

In Section 1, we give the proof of our main theorem about constant 2-labellings of cycles $\mathcal{C}_p$. We consider eight particular weighted cycles $\mathcal{C}_p$ with at most 4 different weights, namely z, x, y and t. The following words represent respectively cycles of Type 1–8 (see Figure 2) :

$$zx^{p-1},\ zx^{\frac{p-2}{2}}tx^{\frac{p-2}{2}},\ z(xy)^{\frac{p-1}{2}},\ z(xy)^{\frac{p-2}{2}}x,\ z(xy)^{\frac{p-1}{4}}(yx)^{\frac{p-1}{4}},$$
$$z(xy)^{\frac{p-3}{4}}xx(yx)^{\frac{p-3}{4}},\ z(xy)^{\frac{p-2}{4}}t(yx)^{\frac{p-2}{4}},\ z(xy)^{\frac{p-4}{4}}xtx(yx)^{\frac{p-4}{4}}$$

with $x \neq y$ and $p \geq 2$. Note that the exponents appearing in the representation of cycles must be integers. This implies extra conditions on p depending on the type of $\mathcal{C}_p$. We describe all constant 2-labellings of these cycles.

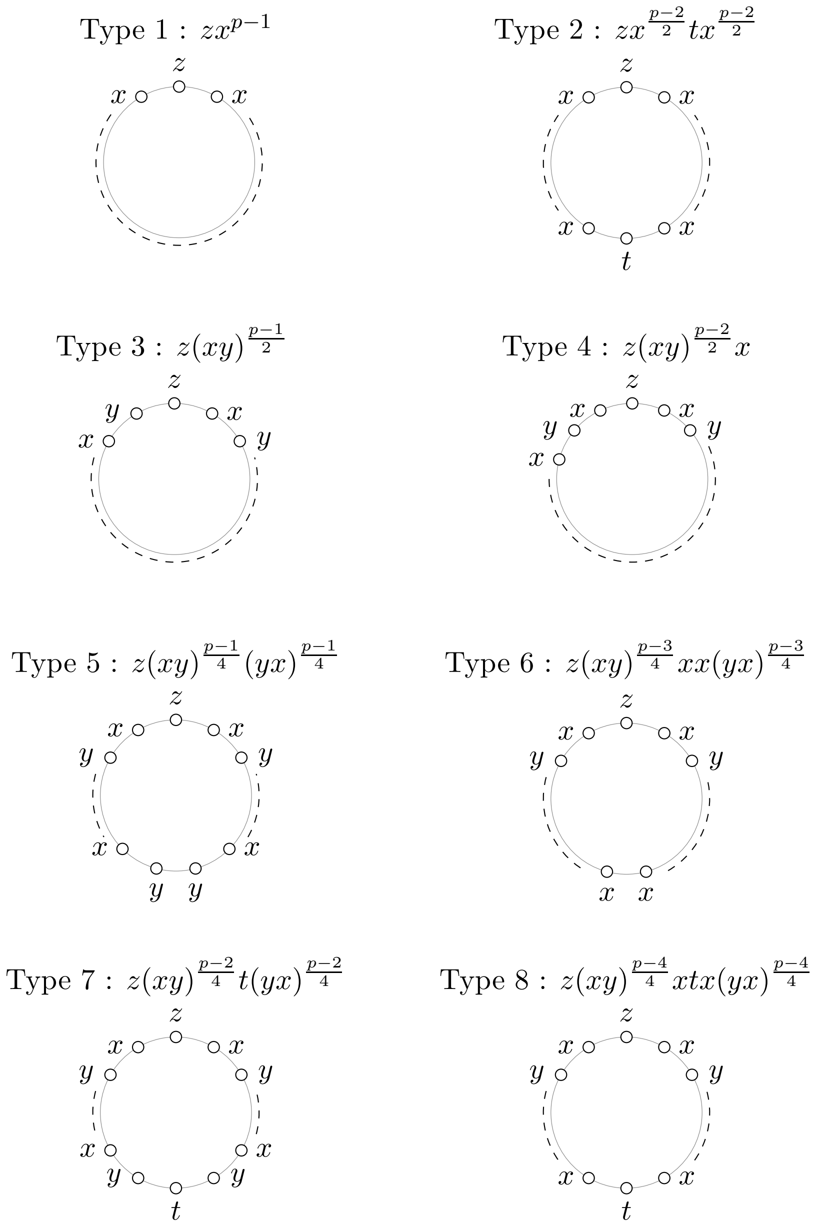


Figure 2: Types of weighted cycles $\mathcal{C}_p$.

Theorem 12. Let φ be a non trivial constant 2-labelling of a cycle $\mathcal{C}_p$ of Type 1–8 with $A = \{\mathcal{R}_k \mid k \in \mathbb{Z}\}$ and $v = 0$. Let $a = \sum_{\{u \in V | \varphi\circ\xi(u)=\bullet\}} w(u)$ and $b = \sum_{\{u \in V | \varphi\circ\xi'(u)=\bullet\}} w(u)$ for $\xi \in A_\bullet, \xi' \in A_\circ$. We have the following possible values of the constants a and b depending of the type of $\mathcal{C}_p$:

Type	Value of a	Value of b	Condition on parameters
1	$\alpha x + z$	$(\alpha+1)x$	$\alpha \in \{0, \dots, p-2\}$
2	$2\alpha x + t + z$	$2(\alpha+1)x$	$\alpha \in \{0, \dots, \frac{p-4}{2}\}$
	$(\frac{p}{2}-1)x + z$	$(\frac{p}{2}-1)x + t$	
3	$\emptyset$	$\emptyset$	
4	$(\alpha+1)x + \alpha y + z$	$(\alpha+1)(x+y)$	$\alpha \in \{0, \dots, \frac{p-4}{2}\}$
	$(\frac{p}{2}-1)y + z$	$\frac{p}{2}x$	
5	$\frac{p}{3}x + (\frac{p}{3}-1)y + z$	$(\frac{p}{3}-1)x + (\frac{p}{3}+1)y$	$p \equiv 0 \pmod 3$
6	$\frac{p}{3}x + (\frac{p}{3}-1)y + z$	$(\frac{p}{3}+1)x + (\frac{p}{3}-1)y$	$p \equiv 0 \pmod 3$
7	$a = (\frac{p}{2}-1)y + z$	$b = (\frac{p}{2}-1)x + t$	
	$a = \alpha(x+y) + t + z$	$b = (\alpha+1)(x+y)$	$\alpha \in \{0, \dots, \frac{p}{2}-1\}$
8	$a = (\frac{p}{2}-2)y + z + t$	$b = \frac{p}{2}x$	
	$a = (2\alpha+2)x + 2\alpha y + z + t$	$b = (2\alpha+2)(x+y)$	$\alpha \in \{0, \dots, \frac{p}{4}-1\}$
	$a = \frac{p}{4}x + (\frac{p}{4}-1)y + z$	$b = \frac{p}{4}x + (\frac{p}{4}-1)y + t$	
	$a = \frac{p}{2}x + (\frac{p}{4}-1)y + z$	$b = \frac{3p}{4}x$	$t = \frac{p}{4}x + (1-\frac{p}{4})y$
	$a = (\frac{p}{4}-1)y + z$	$b = \frac{p}{4}x$	$t = \frac{p}{4}x + (1-\frac{p}{4})y$

In Section 2, we show how Theorem 12 can be useful to solve covering problems in the graph of the infinite grid $\mathbb{Z}^2$. We can view an (r, a, b)-code of $\mathbb{Z}^2$ as a particular coloring of the vertices of $\mathbb{Z}^2$ in black and white such that a ball of radius r centered on a black (resp. white) vertex contain exactly a (resp. b) black vertices, where the balls are defined relative to the Manhattan metric. Finally, we describe all (r, a, b)-codes of $\mathbb{Z}^2$ with $|a - b| > 4$ and $r \geq 2$.

Theorem 16. Let $r, a, b \in \mathbb{N}$ such that $|a - b| > 4$ and $r \geq 2$. For all (r, a, b)-codes of $\mathbb{Z}^2$, the values of a and b are given in the following table.

	a	b	Condition on parameters
Coloring 1			
r even	$r+1+\alpha(2r+1)$	$(\alpha+1)(2r+1)$	$\alpha \in \{0,\dots,r-1\}$
r odd	$3r+2+\alpha(2r+1)$	$(\alpha+1)(2r+1)$	$\alpha \in \{0,\dots,r-2\}$
Coloring 2			
r even	$\frac{r}{2}(2r+1)$	$\frac{r^2}{2}+(\frac{r}{2}+1)(r+1)$	
r odd	$\frac{r+1}{2}(2r+1)$	$\frac{(r+1)^2}{2}+(\frac{r+1}{2}-1)r$	
Coloring 3			
r even	$2(\alpha+1)r+(2\alpha+3)(r+1)$	$2(\alpha+1)(2r+1)$	$\alpha \in \{0,\dots,\frac{r-4}{2}\}$
r even	$(r+1)^2$	r^2	
r odd	$2(\alpha+1)(r+1)+(2\alpha+1)r$	$2(\alpha+1)(2r+1)$	$\alpha \in \{0,\dots,\frac{r-3}{2}\}$
r odd	r^2	$(r+1)^2$	
Coloring 4			
	$\frac{(2r+1)^2}{3}$	$\frac{(2r+1)^2}{3}+1$	$2r+1 \equiv 0 \pmod 3$
Coloring 5			
r even	$(r+1)^2$	r^2	
r odd	r^2	$(r+1)^2$	
$r=3k+1$	$\frac{2r^2+2r-1}{3}+\alpha\frac{2r^2+2r+2}{3}$	$(\alpha+1)\frac{2r^2+2r+2}{3}$	$\alpha \in \{0,1\}$
$r=3k-1$	$\frac{2r^2+2r}{3}-2k+1+\alpha\frac{2r^2+2r}{3}+k$	$(\alpha+1)\frac{2r^2+2r}{3}+k$	$\alpha \in \{0,1\}$
$r=3k$	$\frac{2r^2+2r}{3}+2k-1+\alpha\frac{2r^2+2r}{3}-k$	$(\alpha+1)\frac{2r^2+2r}{3}-k$	$\alpha \in \{0,1\}$

1 Constant 2-labellings

First of all, we introduce basic definitions. For $m \in \mathbb{N}$, we say that a coloring $\varphi : \{0,\dots,p-1\} \to \{\bullet,\circ\}$ of $\mathcal{C}_p$ is

- m-*periodic* if $\varphi(i)=\varphi(j)$ and $j \equiv i+m \pmod p$ for all $i \in \{0,\dots,p-1\}$,
- m-*antiperiodic* if $\varphi(i)\neq\varphi(j)$ and $j \equiv i+m \pmod p$ for all $i \in \{0,\dots,p-1\}$.

If φ is m-periodic, then we call *pattern period* any pattern of length m appearing in the coloring. In particular, a monochromatic coloring of $\mathcal{C}_p$ can be seen as a 1-periodic coloring with pattern period $\varphi(0)$. A coloring φ of $\mathcal{C}_p$ is called *alternate* if φ is 2-periodic and of pattern period $\bullet\circ$ and p is even. In other words, an alternate coloring is 1-antiperiodic and more generally any m-antiperiodic coloring is $2m$-periodic.

Secondly, we give the general scheme of the following lemmas proofs. Let $\mathcal{C}_p$ be a cycle with p weighted vertices and $w : \{0, \ldots, p-1\} \to \mathbb{R}$ be the weight function. Let φ be a constant 2-labelling of $\mathcal{C}_p$ with $A = \{\mathcal{R}_k \mid k \in \mathbb{Z}\}$ and $v = 0$. Recall that

$$a := \sum_{\{u \in V \mid \varphi \circ \xi(u) = \bullet\}} w(u) \text{ and } b := \sum_{\{u \in V \mid \varphi \circ \xi'(u) = \bullet\}} w(u) \quad \text{for } \xi \in A_\bullet, \xi' \in A_\circ.$$

Observe that for a cycle $\mathcal{C}_p$ with at most three different weights, say z, x and y with $x \neq y$ (the case with only weights z, x and t with $x \neq t$ is similar), if the number n of black vertices and the values $a := z + \alpha_x x + \alpha_y y$ and $b := \beta_x x + \beta_y y$ are known, then the following system

$$\begin{cases} a = \lambda x + \mu y + z \\ n = \lambda + \mu + 1 \end{cases} \qquad \left(\text{resp.} \begin{cases} b = \lambda x + \mu y \\ n = \lambda + \mu \end{cases}\right) \tag{1}$$

has a unique solution $(\lambda, \mu) = (\alpha_x, \alpha_y)$ (resp. $(\lambda, \mu) = (\beta_x, \beta_y)$). This means that given the values a and b in terms of z, x and y, we know exactly the number of black vertices of weight x and the number of those of weight y. This will be useful while considering different configurations appearing in the coloring. Therefore, for one particular configuration with $\varphi(0) = \bullet$ (resp. with $\varphi(0) = \circ$), we define $\alpha_x, \alpha_y, \alpha_t$ (resp. $\beta_x, \beta_y, \beta_t$) as the number of black vertices with weight x, y and t. Depending on the type of $\mathcal{C}_p$, we will focus on one of the three different configurations :

$$\langle \varphi(0)\varphi(1) \rangle, \ \left\langle \varphi(0)\varphi(1), \varphi\left(\left\lfloor \frac{p}{2} \right\rfloor + 1\right) \right\rangle, \ \left\langle \varphi(0)\varphi(1), \varphi\left(\left\lfloor \frac{p}{2} \right\rfloor + 1\right) \varphi\left(\left\lfloor \frac{p}{2} \right\rfloor\right) \right\rangle.$$

Note that for cycles of Type 1-6, φ will denote the coloring of $\mathcal{C}_p$ or its rotations. We do not have to make a distinction since the system (1) has a unique solution.

It is easy to check whether or not the alternate coloring is a constant 2-labelling of $\mathcal{C}_p$. Hence we can suppose that φ is not the alternate coloring and, up to complementary, we may assume that we have two consecutive black vertices. Thus we start by considering configuration with $\varphi(0)\varphi(1) = \bullet\bullet$.

For cycles of Types $3-6$, the effect of a 1-rotation is to send vertices of weight x on those of weight y and conversely. Moreover there are possibly some "side effects" around the vertices 0 and $\left\lfloor \frac{p}{2} \right\rfloor \pm 1$ to take into account. Hence, considering configurations with $\varphi(0)\varphi(1) = \bullet\bullet$ and doing a 1-rotation, we obtain a relation between α_x and α_y. For example, on a cycle of Type 4 (see Figure 3), we get $a = \alpha_x x + \alpha_y y + z = (\alpha_x + 1)x + (\alpha_x - 1)y + z$ and so $\alpha_x = \alpha_y + 1$. Similarly, for cycles of Type 2, we have a relation between α_x and α_t.

Since we consider non monochromatic colorings, we can find a configuration with $\varphi(0)\varphi(1) = \bullet\circ$. A 1-rotation gives us the value of the constant b. To conclude, we check whenever the configurations with $\varphi(0)\varphi(1) = \circ\circ$ or $\varphi(0)\varphi(1) = \circ\bullet$ can occur. In some cases, we can show that some patterns are forbidden in a constant 2-labelling.

Now for cycles of Type 7 and 8 with weights z, x, y, t, if the number n of black vertices and the values $a := z + \alpha_x x + \alpha_y y + \alpha_t t$ and $b := \beta_x x + \beta_y y + \beta_t t$ are known, then the

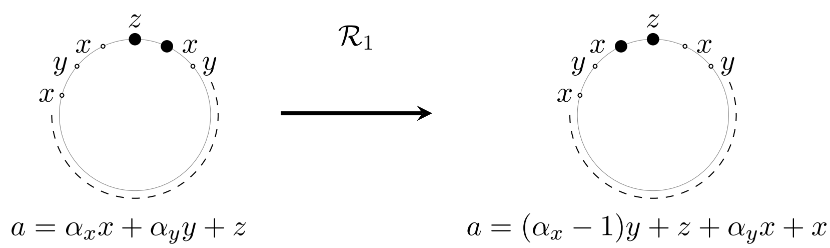


Figure 3: A 1-rotation of a coloring φ with $\varphi(0)\varphi(1) = \bullet\bullet$ for cycles of Type 4.

following system

$$\begin{cases} a = \lambda x + \mu y + \nu t + z \\ n = \lambda + \mu + \nu + 1 \end{cases} \qquad \left(\text{respectively } \begin{cases} b = \lambda x + \mu y + \nu t \\ n = \lambda + \mu + \nu \end{cases}\right)$$

doesn't neccessarily have a unique solution $(\lambda, \mu, \nu) = (\alpha_x, \alpha_y, \alpha_t)$ (resp. $(\lambda, \mu, \nu) = (\beta_x, \beta_y, \beta_t)$). Hence in this case, it will be important to make a disctinction between the coloring φ and its rotations. Knowing the configuration $\left\langle \varphi(0)\varphi(1), \varphi\left(\left\lfloor \frac{p}{2} \right\rfloor + 1\right) \varphi\left(\left\lfloor \frac{p}{2} \right\rfloor\right)\right\rangle$, we can deduce the configuration of $\varphi \circ \mathcal{R}_1$ and so on.

The proof of the main theorem can be done by the following lemmas.

Lemma 4. *For cycles $\mathcal{C}_p$ of Type 1, i.e., zx^{p-1} with $1 < p \in \mathbb{N}$, all colorings are constant 2-labellings.*

Proof. Let φ be a coloring of $\mathcal{C}_p$. If the vertex 0 is black, we set α_x the number of black vertices with weight x. So we have $\alpha_x + 1$ black vertices and it is clear that φ is a constant 2-labelling where the weighted sum of black vertices is equal to $a = \alpha_x x + z$ (resp. $b = (\alpha_x + 1)x$) if the vertex 0 is black (resp. white). $\square$

Lemma 5. *For cycles $\mathcal{C}_p$ of Type 2, i.e., $zx^{\frac{p-2}{2}}tx^{\frac{p-2}{2}}$ with $2 < p \in \mathbb{N}$ and $x \neq t$, if φ is a non trivial constant 2-labelling of $\mathcal{C}_p$, then φ is either $\frac{p}{2}$-periodic with $a = 2\alpha x + t + z$ and $b = 2(\alpha + 1)x$ for any $\alpha \in \{0, \ldots, \frac{p-4}{2}\}$ or $\frac{p}{2}$-antiperiodic with $a = (\frac{p}{2} - 1)x + z$ and $b = (\frac{p}{2} - 1)x + t$.*

Proof. Note that the number p of vertices of a cycle $\mathcal{C}_p$ of Type 2 is even. It is clear that the alternate coloring is a constant 2-labelling. If $p \equiv 0 \pmod 4$, we have $a = (\frac{p}{2} - 2)x + t + z$, $b = \frac{p}{2}x$ and the coloring is $\frac{p}{2}$-periodic. In particular, it corresponds to the second case of the statement with $\alpha = \frac{p-1}{4}$. If $p \equiv 2 \pmod 4$, we have $a = (\frac{p}{2} - 1)x + z$, $b = (\frac{p}{2} - 1)x + t$ and the coloring is $\frac{p}{2}$-antiperiodic which corresponds to the third case of the statement.

Now, let φ be a non trivial constant 2-labelling which is not the alternate coloring. We need to consider configurations of type $\left\langle \varphi(0)\varphi(1), \varphi(\frac{p}{2} + 1)\right\rangle$. Assume $\varphi(0) = \bullet$ and let α_x and α_t be defined as before. We have $a = \alpha_x x + \alpha_t t + z$. The values of the constant a or

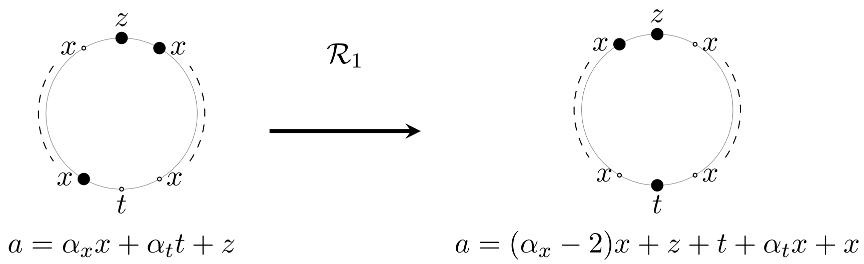


Figure 4: A 1-rotation of a coloring with configuration $\langle \bullet\bullet, \bullet \rangle$ for a Type 2 cycle.

b obtained after one rotation of a configuration $\left\langle \bullet\varphi(1), \varphi(\frac{p}{2}+1) \right\rangle$ are given in the following table (see Figure 4 for an illustration of the first line in the table).

Configuration	Value of the constant after a 1-rotation
$\langle \bullet\bullet, \bullet \rangle$	$a = (\alpha_x - 2)x + z + t + \alpha_t x + x = (\alpha_t + \alpha_x - 1)x + t + z$
$\langle \bullet\bullet, \circ \rangle$	$a = (\alpha_x - 1)x + z + \alpha_t x + x = (\alpha_x + \alpha_t)x + z$
$\langle \bullet\circ, \bullet \rangle$	$b = (\alpha_x - 1)x + t + \alpha_t x + x = (\alpha_x + \alpha_t)x + t$
$\langle \bullet\circ, \circ \rangle$	$b = \alpha_x x + \alpha_t x + x = (\alpha_x + \alpha_t + 1)x$

Therefore as $x \neq t$, the configurations $\langle \bullet\bullet, \bullet \rangle$, $\langle \bullet\bullet, \circ \rangle$ (resp.$\langle \bullet\circ, \bullet \rangle$ and $\langle \bullet\circ, \circ \rangle$) can not both appear in the coloring since they imply different values for the constant a (resp. b). Note that, if $\langle \bullet\bullet, \bullet \rangle$ (resp. $\langle \bullet\bullet, \circ \rangle$) appears in the constant 2-labelling, then we have $\alpha_t = 1$ (resp. $\alpha_t = 0$) as $x \neq t$. Hence we have four possible cases to consider.

Case 1.A : Both configurations $\langle \bullet\bullet, \bullet \rangle$, $\langle \bullet\circ, \bullet \rangle$ appear in φ. We know that $a = \alpha_x x + t + z$ and $b = (\alpha_x + 1)x + t$ with $0 \leq \alpha_x \leq p - 3$. The configurations $\langle \circ\bullet, \circ \rangle$ and $\langle \circ\circ, \circ \rangle$ are forbidden since they imply respectively that $a = \alpha_x x + z + x$ and $b = (\alpha_x + 1)x + x$ which is a contradiction.

So only the configurations $\langle \bullet\bullet, \bullet \rangle$, $\langle \bullet\circ, \bullet \rangle$, $\langle \circ\bullet, \bullet \rangle$ and $\langle \circ\circ, \bullet \rangle$ possibly appear in the coloring. Observe that it implies that $\varphi(\frac{p}{2}+1) = \bullet$ for all colorings of $\varphi(0)\varphi(1)$. Hence φ is trivial which is a contradiction.

Case 1.B : Both configurations $\langle \bullet\bullet, \bullet \rangle$, $\langle \bullet\circ, \circ \rangle$ appear in φ. We have that $a = \alpha_x x + t + z$ and $b = (\alpha_x + 2)x$ with $0 \leq \alpha_x \leq p - 3$. The configurations $\langle \circ\bullet, \circ \rangle$ and $\langle \circ\circ, \bullet \rangle$ give different values for a and b. We get respectively $a = (\alpha_x + 1)x + z$ and $b = (\alpha_x + 1)x + t$ which is a contradiction.

So only the configurations $\langle \bullet\bullet, \bullet \rangle$, $\langle \bullet\circ, \circ \rangle$,$\langle \circ\bullet, \bullet \rangle$ and $\langle \circ\circ, \circ \rangle$ possibly appear in the coloring. It means that diametrically opposed vertices have always the same color. Thus φ is $\frac{p}{2}$-periodic and the number $\alpha_x + 2$ of black vertices is even. We have $a = 2\alpha x + t + z$ and $b = 2(\alpha + 1)x$ for $\alpha \in \{0, \ldots, \frac{p-4}{2}\}$.

Case 2.A : Both configurations $\langle \bullet\bullet, \circ \rangle$, $\langle \bullet\circ, \bullet \rangle$ appear in φ. We know that $a = \alpha_x x + z$

and $b = \alpha_x x + t$ with $0 \leq \alpha_x \leq p-2$. The configurations $\langle \circ\bullet, \bullet \rangle$ and $\langle \circ\circ, \circ \rangle$ are impossible as they give respectively by 1-rotation $a = (\alpha_x - 2)x + t + z + x = (\alpha_x - 1)x + y + z \neq a$ and $b = \alpha_x x + x = (\alpha_x + 1)x \neq b$.

So only the configurations $\langle \bullet\bullet, \circ \rangle$, $\langle \bullet\circ, \bullet \rangle$ $\langle \circ\bullet, \circ \rangle$ and $\langle \circ\circ, \bullet \rangle$ possibly appear. This implies that diametrically opposed vertices are of different colors and φ is $\frac{p}{2}$-antiperiodic. In particular, this means that the number of black (resp. white) vertices is exactly $\frac{p}{2}$. Hence, $a = (\frac{p}{2} - 1)x + z$ and $b = (\frac{p}{2} - 1)x + t$.

<u>Case 2.B</u> : Both configurations $\langle \bullet\bullet, \circ \rangle$, $\langle \bullet\circ, \circ \rangle$ appear in φ. We have $a = \alpha_x x + z$ and $b = (\alpha_x + 1)x$ with $0 \leq \alpha_x \leq p-2$. Using the same argument as in Case 1.A, we can show that $\langle \circ\bullet, \bullet \rangle$ and $\langle \circ\circ, \bullet \rangle$ can not appear in the coloring as they imply respectively $a = (\alpha_x - 1)x + z + t$ and $b = \alpha_x x + t$.

So only the configurations $\langle \bullet\bullet, \circ \rangle$, $\langle \bullet\circ, \circ \rangle$, $\langle \circ\bullet, \circ \rangle$ and $\langle \circ\circ, \circ \rangle$ can appear in the φ. Hence φ is trivial which is a contradiction. □

Lemma 6. *For cycles $\mathcal{C}_p$ of Type 3, i.e., $z(xy)^{\frac{p-1}{2}}$ with $x \neq y$ and $1 < p \in \mathbb{N}$, only monochromatic colorings are constant 2-labellings.*

Proof. As the number of vertices is odd, the alternate coloring is impossible. So let φ be a non trivial constant 2-labelling which is not the alternate coloring. We consider the configuration $\langle \varphi(0)\varphi(1) \rangle = \langle \bullet\bullet \rangle$. By definition of α_x and α_y, we have :

$$a = \alpha_x x + \alpha_y y + z.$$

After a rotation, we get $a = (\alpha_x - 1)y + z + \alpha_y x + y$. So we have $(\alpha_x - \alpha_y)(x - y) = 0$. Hence, $\alpha_x = \alpha_y$ as $x \neq y$. Since φ is not trivial, there is a white vertex after some blacks and we have the configuration $\langle \bullet\circ \rangle$ after some rotations. We know that $a = \alpha_x x + \alpha_x y + z$. By rotation, we get

$$b = \alpha_x y + \alpha_x x + y = \alpha_x x + (\alpha_x + 1)y.$$

Observe that if we have the configuration $\langle \circ\circ \rangle$, then by rotation we get $b = \alpha_x y + (\alpha_x + 1)x \neq \alpha_x x + (\alpha_x + 1)y = b$ as $x \neq y$. Using the same arguments, the configuration $\langle \circ\bullet \rangle$ implies that $a = (\alpha_x - 1)y + z + (\alpha_x + 1)x \neq \alpha_x x + \alpha_x y + z = a$. So it is impossible to have a white vertex followed by a white or by a black. Hence there is no possible non trivial constant 2-labelling. □

Lemma 7. *For cycles $\mathcal{C}_p$ of Type 4, i.e., $z(xy)^{\frac{p-2}{2}}x$ with $x \neq y$ and $2 < p \in \mathbb{N}$, if φ is a non trivial constant 2-labelling, then φ is either the alternate coloring with $a = (\frac{p}{2} - 1)y + z$ and $b = \frac{p}{2}x$, or any coloring such that the number of black vertices of weight x and of weight y are equal when $\varphi(0) = \circ$. In the last case, we have $a = (\alpha + 1)x + \alpha y + z$ and $b = (\alpha + 1)(x + y)$ for any $\alpha \in \{0, \ldots, \frac{p-4}{2}\}$.*

Proof. By definition of Type 4, p is even and the alternate coloring is a constant 2-labelling with $a = (\frac{p}{2} - 1)y + z$ and $b = \frac{p}{2}x$.

Now assume that φ is a non trivial constant 2-labelling of $\mathcal{C}_p$ which is not the alternate coloring and consider the configuration $\langle \varphi(0)\varphi(1) \rangle = \langle \bullet\bullet \rangle$. By definition of α_x and α_y, we have $a = \alpha_x x + \alpha_y y + z$. After a 1-rotation, we have $a = (\alpha_x - 1)y + z + \alpha_y x + x = (\alpha_y + 1)x + (\alpha_x - 1)y + z$. So we get $\alpha_x = \alpha_y + 1$. Since φ is not trivial, the configuration $\langle \bullet\circ \rangle$ appears in the coloring. As $a = (\alpha_y + 1)x + \alpha_y y + z$, we have by a 1-rotation $b = (\alpha_y + 1)y + (\alpha_y + 1)x$. The configurations $\langle \circ\circ \rangle$ and $\langle \circ\bullet \rangle$ are also possible. Indeed, they give by 1-rotation the same values of a and b as before.

Hence φ must be such that, if $\varphi(0) = \bullet$, there is one more black vertex of weight x than of weight y. We get $a = (\alpha + 1)x + \alpha y + z$ and $b = (\alpha + 1)(x + y)$ for $\alpha \in \{0, \ldots, \frac{p-4}{2}\}$. $\square$

Lemma 8. *For cycles $\mathcal{C}_p$ of Type 5, i.e., $z(xy)^{\frac{p-1}{4}}(yx)^{\frac{p-1}{4}}$ with $x \neq y$ and $1 < p \in \mathbb{N}$, if φ is a non trivial constant 2-labelling, then $p \equiv 0 \pmod 3$ and φ is 3-periodic of pattern period $\bullet\bullet\circ$.*

Proof. By definition of Type 5, the number p of vertices is such that $p \equiv 1 \pmod 4$. Hence the alternate coloring is impossible. Let φ be a non trivial constant 2-labelling of $\mathcal{C}_p$ which is not the alternate coloring and consider configurations of type $\left\langle \varphi(0)\varphi(1), \varphi(\lfloor \frac{p}{2} \rfloor + 1) \right\rangle$. Assume $\varphi(0) = \bullet$ and let α_x and α_y be defined as before. We have $a = \alpha_x x + \alpha_y y + z$. The following table gives the value of the constant a or b depending on the different configurations.

Configuration	Value of the constant after a 1-rotation
$\langle \bullet\bullet, \bullet \rangle$	$a = (\alpha_x - 1)y + z + (\alpha_y - 1)x + y + x = \alpha_y x + \alpha_x y + z$
$\langle \bullet\bullet, \circ \rangle$	$a = (\alpha_x - 1)y + z + \alpha_y x + x = (\alpha_y + 1)x + (\alpha_x - 1)y + z$
$\langle \bullet\circ, \bullet \rangle$	$b = \alpha_x y + (\alpha_y - 1)x + y + x = \alpha_y x + (\alpha_x + 1)y$
$\langle \bullet\circ, \circ \rangle$	$b = \alpha_x y + \alpha_y x + x = (\alpha_y + 1)x + \alpha_x y$

Therefore, the configurations $\langle \bullet\bullet, \bullet \rangle$, $\langle \bullet\bullet, \circ \rangle$ (respectively $\langle \bullet\circ, \bullet \rangle$ and $\langle \bullet\circ, \circ \rangle$) can not both appear in the coloring since they imply different values for the constant a (resp. b). Note that, if $\langle \bullet\bullet, \bullet \rangle$ (respectively $\langle \bullet\bullet, \circ \rangle$) appears in the constant 2-labelling, then we have $\alpha_x = \alpha_y$ (resp. $\alpha_x = \alpha_y + 1$) as $x \neq y$. Hence we have four possible cases to consider.

<u>Case 1.A</u> : Both configurations $\langle \bullet\bullet, \bullet \rangle$, $\langle \bullet\circ, \bullet \rangle$ appear in φ. We know that $a = \alpha_x x + \alpha_x y + z$ and $b = \alpha_x x + (\alpha_x + 1)y$ with $0 \leq \alpha_x < \frac{p-1}{2}$. The configurations $\langle \circ\bullet, \circ \rangle$ and $\langle \circ\circ, \circ \rangle$ give different values for a and b. We have respectively $a = (\alpha_x - 1)y + z + (\alpha_x + 1)x$ and $b = \alpha_x y + (\alpha_x + 1)x$ which is a contradiction. Hence they do not appear in φ.

So only the configurations $\langle \bullet\bullet, \bullet \rangle$, $\langle \bullet\circ, \bullet \rangle$, $\langle \circ\bullet, \bullet \rangle$ and $\langle \circ\circ, \bullet \rangle$ possibly appear in φ. As $\varphi(\frac{p}{2} + 1) = \bullet$ for all colorings of $\varphi(0)\varphi(1)$, this implies that φ is trivial which is a contradiction.

<u>Case 1.B</u> : Both configurations $\langle \bullet\bullet, \bullet \rangle$, $\langle \bullet\circ, \circ \rangle$ appear in φ. We have that $a = \alpha_x x + \alpha_x y + z$ and $b = (\alpha_x + 1)x + \alpha_x y$ with $0 \leq \alpha_x < \frac{p-1}{2}$. The configurations $\langle \circ\bullet, \bullet \rangle$, $\langle \circ\circ, \bullet \rangle$ and $\langle \circ\circ, \circ \rangle$ give different values for a and b. Indeed, we get respectively $a = \alpha_x y + z + (\alpha_x - 1)x + y$, $b = (\alpha_x + 1)y + (\alpha_x - 1)x + y = (\alpha_x - 1)x + (\alpha_x + 2)y$ and $b = (\alpha_x + 1)y + \alpha_x x$ which is

a contradiction. In particular, it means that the pattern $\circ\circ$ is forbidden. Hence, a white vertex is always followed by a black.

Using the only possible configurations $\langle\bullet\bullet,\bullet\rangle$, $\langle\bullet\circ,\circ\rangle$ and $\langle\circ\bullet,\circ\rangle$ appearing in φ, we can deduce that the pattern $\circ\bullet\circ$ is forbidden. Indeed, if $\varphi(0)\varphi(1)\varphi(2) = \circ\bullet\circ$, then $\varphi(\lfloor\frac{p}{2}\rfloor+1)\varphi(\lfloor\frac{p}{2}\rfloor+2) = \circ\circ$ which is a contradiction.

Moreover, the pattern $\circ\bullet\bullet$ is forbidden. If $\varphi(0)\varphi(1)\varphi(2) = \circ\bullet\bullet$, then we get $\varphi(\lfloor\frac{p}{2}\rfloor+1) = \circ$ and $\varphi(\lfloor\frac{p}{2}\rfloor+2) = \bullet$. After a $\lfloor\frac{p}{2}\rfloor$-rotation, we obtain, as described at Figure 5, the configuration $\langle\circ\bullet,\bullet\rangle$ which is impossible.

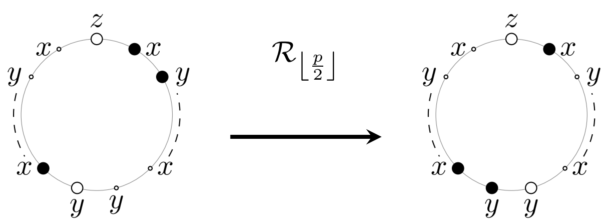


Figure 5: A $\lfloor\frac{p}{2}\rfloor$-rotation of a coloring of a Type 5 cycle $\mathcal{C}_p$.

Since the patterns $\circ\circ$, $\circ\bullet\circ$ and $\circ\bullet\bullet$ are forbidden, no white vertex can appear. Hence φ is trivial which is a contradiction.

<u>Case 2.A</u> : Both configurations $\langle\bullet\bullet,\circ\rangle$, $\langle\bullet\circ,\bullet\rangle$ appear in φ. We know that $a = (\alpha_y+1)x+\alpha_y y+z$ and $b = \alpha_y x+(\alpha_y+2)y$ with $0 \le \alpha_y < \frac{p-2}{2}$. The configurations $\langle\circ\bullet,\circ\rangle$, $\langle\circ\circ,\bullet\rangle$ and $\langle\circ\circ,\circ\rangle$ are impossible as they give respectively different values of a and b by 1-rotation:

$$a = (\alpha_y-1)y+z+(\alpha_y+2)x,\ b = \alpha_y y+x+(\alpha_y+1)x+y \text{ and } b = \alpha_y y+(\alpha_y+2)x.$$

In particular, the pattern $\circ\circ$ is forbidden.

Using the same arguments as in the previous case, we can show that if the pattern $\bullet\bullet\bullet$ appears, then it implies that the pattern $\circ\circ$ occurs in φ. Also, if the pattern $\circ\bullet\circ$ appears, then we can get a contradiction using a $\lfloor\frac{p}{2}\rfloor$-rotation.

So if $p \not\equiv 0 \pmod 3$, then φ is trivial which is a contradiction. Otherwise, $p \equiv 0 \pmod 3$ and φ is 3-periodic of pattern period $\bullet\bullet\circ$. In this case, we have $a = \frac{p}{3}x+(\frac{p}{3}-1)y+z$ and $b = (\frac{p}{3}-1)x+(\frac{p}{3}+1)y$.

<u>Case 2.B</u> : Both configurations $\langle\bullet\bullet,\circ\rangle$, $\langle\bullet\circ,\circ\rangle$ appear in φ. We have $a = (\alpha_y+1)x+\alpha_y y+z$, $b = (\alpha_y+1)x+(\alpha_y+1)y$ with $0 \le \alpha_y < \frac{p-2}{2}$. We can show that $\langle\circ\bullet,\bullet\rangle$ and $\langle\circ\circ,\bullet\rangle$ can not appear in φ as they imply respectively $a = \alpha_y y+z+\alpha_y x+y$ and $b = (\alpha_y+1)y+\alpha_y x+y$ which is a contradiction.

So only the configurations $\langle\bullet\bullet,\circ\rangle$, $\langle\bullet\circ,\circ\rangle$, $\langle\circ\bullet,\circ\rangle$ and $\langle\circ\circ,\circ\rangle$ can appear in φ. Hence φ is trivial which is a contradiction. $\square$

Lemma 9. *For cycles $\mathcal{C}_p$ of Type 6, i.e., $z(xy)^{\frac{p-3}{4}}xx(yx)^{\frac{p-3}{4}}$ with $x \neq y$ and $3 < p \in \mathbb{N}$, if φ is a non trivial constant 2-labelling, then $p \equiv 0 \pmod 3$ and φ is 3-periodic of pattern period $\bullet\bullet\circ$.*

The proof follows exactly the same lines as the proof of Lemma 8.

Proof. By definition of Type 6, $p \equiv 3 \pmod 4$ and the alternate coloring is impossible. Let φ be a non trivial constant 2-labelling of $\mathcal{C}_p$ which is not the alternate coloring and consider configurations of type $\langle \varphi(0)\varphi(1), \varphi(\lfloor \frac{p}{2} \rfloor + 1)\rangle$. Assume $\varphi(0) = \bullet$ and let α_x and α_y be defined as before. We have $a = \alpha_x x + \alpha_y y + z$. The following table gives the value of the constant a or b depending on the different configurations.

Configuration	Value of the constant after a 1-rotation
$\langle \bullet\bullet, \bullet\rangle$	$a = (\alpha_x - 2)y + z + x + \alpha_y x + x = (\alpha_y + 2)x + (\alpha_x - 2)y + z$
$\langle \bullet\bullet, \circ\rangle$	$a = (\alpha_x - 1)y + z + \alpha_y x + x = (\alpha_y + 1)x + (\alpha_x - 1)y + z$
$\langle \bullet\circ, \bullet\rangle$	$b = (\alpha_x - 1)y + x + \alpha_y x + x = (\alpha_y + 2)x + (\alpha_x - 1)y$
$\langle \bullet\circ, \circ\rangle$	$b = \alpha_x y + \alpha_y x + x = (\alpha_y + 1)x + \alpha_x y$

Therefore, the configurations $\langle \bullet\bullet, \bullet\rangle$, $\langle \bullet\bullet, \circ\rangle$ (respectively $\langle \bullet\circ, \bullet\rangle$ and $\langle \bullet\circ, \circ\rangle$) can not both appear in the coloring since they imply different values for the constant a (resp. b). Note that, if $\langle \bullet\bullet, \bullet\rangle$ (respectively $\langle \bullet\bullet, \circ\rangle$) appears in the constant 2-labelling, then we have $\alpha_x = \alpha_y + 2$ (resp. $\alpha_x = \alpha_y + 1$) as $x \neq y$. Hence we have four possible cases to consider.

Case 1.A : Both configurations $\langle \bullet\bullet, \bullet\rangle$, $\langle \bullet\circ, \bullet\rangle$ appear in φ. We have $a = (\alpha_y + 2)x + \alpha_y y + z$ and $b = (\alpha_y + 2)x + (\alpha_y + 1)y$ with $0 \leq \alpha_y < \frac{p-3}{2}$. It is easy to see by rotation that the configurations $\langle \circ\bullet, \circ\rangle$ and $\langle \circ\circ, \circ\rangle$ can not appear in φ. So the only possible configurations are $\langle \bullet\bullet, \bullet\rangle$, $\langle \bullet\circ, \bullet\rangle$, $\langle \circ\bullet, \bullet\rangle$ and $\langle \circ\circ, \bullet\rangle$. Hence φ is trivial which is a contradiction.

Case 1.B : Both configurations $\langle \bullet\bullet, \bullet\rangle$, $\langle \bullet\circ, \circ\rangle$ appear in φ. We know that $a = (\alpha_y+2)x + \alpha_y y + z$ and $b = (\alpha_y + 1)x + (\alpha_y + 2)y$ with $0 \leq \alpha_y < \frac{p-3}{2}$. The configurations $\langle \circ\bullet, \bullet\rangle$, $\langle \circ\circ, \bullet\rangle$ and $\langle \circ\circ, \circ\rangle$ give different values for a and b after rotation. They are thus impossible. So the pattern $\circ\circ$ is forbidden. Using the possible configurations, we can deduce that the pattern $\circ\bullet\circ$ is forbidden as it implies an occurence of the pattern $\circ\circ$. By a $\lfloor \frac{p}{2} \rfloor$-rotation, the pattern $\circ\bullet\bullet$ is also forbidden. Hence the constant 2-labelling is trivial which is a contradiction.

Case 2.A : Both configurations $\langle \bullet\bullet, \circ\rangle$, $\langle \bullet\circ, \bullet\rangle$ appear in φ. We have $a = (\alpha_y + 1)x + \alpha_y y + z$ and $b = (\alpha_y + 2)x + \alpha_y y$ with $0 \leq \alpha_y < \frac{p-2}{2}$. The configurations $\langle \circ\bullet, \circ\rangle$, $\langle \circ\circ, \bullet\rangle$ and $\langle \circ\circ, \circ\rangle$ are impossible. So the patterns $\circ\circ$ and $\bullet\bullet\bullet$ are forbidden. By a $\lfloor \frac{p}{2} \rfloor$-rotation, the pattern $\circ\bullet\circ$ is also forbidden. Hence, if $p \not\equiv 0 \pmod 3$, then φ is trivial which is a contradiction. Otherwise, $p \equiv 0 \pmod 3$ and φ must be 3-periodic of pattern period $\bullet\bullet\circ$. In this case, we have $a = \frac{p}{3}x + (\frac{p}{3} - 1)y + z$ and $b = (\frac{p}{3} + 1)x + (\frac{p}{3} - 1)y$.

Case 2.B : Both configurations $\langle \bullet\bullet, \circ\rangle$, $\langle \bullet\circ, \circ\rangle$ appear in φ. We have $a = (\alpha_y + 1)x + \alpha_y y + z$ and $b = (\alpha_y + 1)x + (\alpha_y + 1)y$ with $0 \leq \alpha_y < \frac{p-2}{2}$. The configurations $\langle \circ\bullet, \bullet\rangle$ and $\langle \circ\circ, \bullet\rangle$ can not appear in φ since they give different values for a and b after rotation.

So only the configurations $\langle \bullet\bullet, \circ\rangle$, $\langle \bullet\circ, \circ\rangle$, $\langle \circ\bullet, \circ\rangle$ and $\langle \circ\circ, \circ\rangle$ are possible. Hence φ is trivial which is a contradiction. $\square$

Lemma 10. *For cycles $\mathcal{C}_p$ of Type 7, i.e., $z(xy)^{\frac{p-2}{4}}t(yx)^{\frac{p-2}{4}}$ with $t \neq x \neq y$ and $2 < p \in \mathbb{N}$, if φ is a non trivial constant 2-labelling, then φ is either alternate with $a = (\frac{p}{2}-1)y + z$ and $b = (\frac{p}{2}-1)x + t$ or $\frac{p}{2}$-periodic with $a = \alpha(x+y) + t + z$ and $b = (\alpha+1)(x+y)$ for $\alpha \in \{0, \dots, \frac{p}{2}-1\}$.*

Proof. By definition of Type 7, p is even and the alternate coloring is a constant 2-labelling with $a = (\frac{p}{2}-1)y + z$ and $b = (\frac{p}{2}-1)x + t$. Suppose that φ is a non trivial constant 2-labelling of $\mathcal{C}_p$ which is not the alternate coloring. We consider configurations of type $\left\langle \varphi(0)\varphi(1), \varphi(\frac{p}{2}+1)\varphi(\frac{p}{2})\right\rangle$. Assume that $\varphi(0)\varphi(1) = \bullet\bullet$. We have four cases to consider.

Case 1 : The configuration $\langle \bullet\bullet, \bullet\bullet\rangle$ appears in φ. We have $a = \alpha_x x + \alpha_y y + z + t$ and by rotation, we get $a = (\alpha_x - 1)y + z + (\alpha_y - 1)x + t + x + y$. Hence, $\alpha_x = \alpha_y := \alpha$ and we have $0 \leq \alpha < \frac{p-2}{2}$. Observe that the numbers of black vertices with respective weights x, y, z and t are constant for rotation of colorings with the configuration $\langle \bullet\bullet, \bullet\bullet\rangle$.

Let $i \in \{0, \dots, \frac{p}{2}-1\}$ be the smallest integer such that $\varphi(i+1) = \circ$. As φ is non trivial, such an integer exists. We can assume that

$$\varphi\left(\frac{p}{2} + \ell\right) = \bullet \qquad \forall \ell \in \{0, \dots, i\}.$$

Otherwise, we consider the coloring $\varphi \circ \mathcal{R}_{\frac{p}{2}}$ instead of φ.

It implies that $\varphi\left(\frac{p}{2} + i + 1\right) = \circ$. Indeed, assume that $\varphi\left(\frac{p}{2} + i + 1\right) = \bullet$ as illustrated at Figure 6. Then the coloring $\varphi \circ \mathcal{R}_i$ has configuration $\langle \bullet\circ, \bullet\bullet\rangle$ with $a = \alpha x + \alpha y + z + t$. Since a $\frac{p}{2}$-rotation maps vertices of weight x on vertices of weight y and vice versa, the coloring $\varphi \circ \mathcal{R}_{i+\frac{p}{2}}$ has configuration $\langle \bullet\bullet, \circ\bullet\rangle$ with $a = \alpha x + \alpha y + z + t$. By rotation, we obtain $a = \alpha x + (\alpha - 1)y + z + x + y = (\alpha+1)x + \alpha y + z$. It implies $x = t$ which is a contradiction.

Hence $\varphi(i+1) = \varphi\left(\frac{p}{2} + i + 1\right) = \circ$ and the coloring $\varphi \circ \mathcal{R}_{i+1}$ has a weighted sum of black vertices equal to $b = \alpha y + \alpha x + x + y = (\alpha+1)(x+y)$. Observe that for a coloring with configuration $\langle \circ\circ, \circ\circ\rangle$ and weighted sum $b = (\alpha+1)(x+y)$, we obtain again $b = (\alpha+1)(y+x)$ after rotation.

Let $j \in \{i+1, \dots, \frac{p}{2}-1\}$ be the smallest integer such that $\varphi(j+1) = \bullet$. Without loss of generality, we can assume that

$$\varphi\left(\frac{p}{2} + \ell\right) = \circ \qquad \forall \ell \in \{i+1, \dots, j\}.$$

Therefore $\varphi\left(\frac{p}{2} + j + 1\right) = \bullet$. Indeed, assume that $\varphi\left(\frac{p}{2} + j + 1\right) = \circ$. Then the coloring $\varphi \circ \mathcal{R}_j$ has configuration $\langle \circ\bullet, \circ\circ\rangle$ with $b = (\alpha+1)(x+y)$. By rotation, we get $a = \alpha y + z + (\alpha+1)x$. Hence, $x = t$ which is a contradiction.

So the coloring $\varphi \circ \mathcal{R}_{j+1}$ has configuration

$$\left\langle \bullet\, (\varphi \circ \mathcal{R}_{j+1})\,(1), (\varphi \circ \mathcal{R}_{j+1})\left(\frac{p}{2} + 1\right) \bullet \right\rangle$$

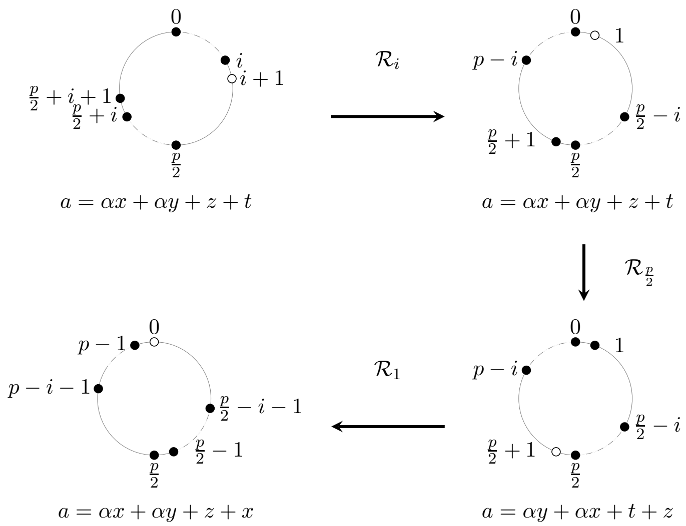


Figure 6: Rotations of a coloring of Type 7 cycle $\mathcal{C}_p$ and their corresponding weighted sums of black vertices.

with $a = \alpha x + \alpha y + z + t$. This is the same situation as in the beginning.

Thus φ is $\frac{p}{2}$-periodic. We have $a = \alpha(x+y) + z + t$ and $b = (\alpha+1)(x+y)$ for $\alpha \in \{0, \ldots, \frac{p}{2} - 2\}$.

Case 2 : The configuration $\langle \bullet\bullet, \circ\circ \rangle$ appears in φ. We have $a = \alpha_x x + \alpha_y y + z$ and by rotation, $a = (\alpha_x - 1)y + z + \alpha_y x + x$. Hence $\alpha_x = \alpha_y + 1$. Set $\alpha := \alpha_y$. We have $a = (\alpha+1)x + \alpha y + z$ with $1 \leq \alpha < \frac{p-2}{2}$. Moreover we know the value of the constant b as the coloring $\varphi \circ \mathcal{R}_{\frac{p}{2}}$ has configuration $\langle \circ\circ, \bullet\bullet \rangle$ with a weighted sum of black vertices equal to $b = (\alpha+1)y + \alpha x + t$.

Let $i \in \{0, \ldots, \frac{p}{2} - 1\}$ be the smallest integer such that $\varphi(i+1) = \circ$. As φ is non trivial, such an integer exists. We can assume that

$$\varphi\left(\frac{p}{2} + \ell\right) = \circ \qquad \forall \ell \in \{0, \ldots, i\}.$$

Otherwise, we consider the coloring $\varphi \circ \mathcal{R}_{\frac{p}{2}}$ instead of φ and we apply the same reasoning with a swapping of the colors.

Assume that $\varphi\left(\frac{p}{2} + i + 1\right) = \circ$. Then the coloring $\varphi \circ \mathcal{R}_i$ has configuration $\langle \bullet\circ, \circ\circ \rangle$ with $a = \alpha x + \alpha y + z + t$. By rotation, we obtain $b = (\alpha+1)y + (\alpha+1)x$. It implies $x = t$ which is a contradiction. Hence $\varphi\left(\frac{p}{2} + i + 1\right) = \bullet$. Observe that for any coloring with configuration $\langle \circ\circ, \bullet\bullet \rangle$ and weighted sum $b = \alpha x + (\alpha+1)y + t$, we obtain again $b = \alpha y + \alpha x + t + y = \alpha x + (\alpha+1)y + t$ after rotation.

Let $j \in \{i+1, \ldots, \frac{p}{2} + i\}$ be the smallest integer such that $\varphi(j+1) = \bullet$. Without loss

of generality, we can assume that

$$\varphi\left(\frac{p}{2}+\ell\right)=\bullet \qquad \forall \ell\in\{i+1,\ldots,j\}.$$

Therefore $\varphi\left(\frac{p}{2}+j+1\right)=\bullet$. Indeed, assume that $\varphi\left(\frac{p}{2}+j+1\right)=\circ$. Then the coloring $\varphi\circ\mathcal{R}_j$ has configuration $\langle\circ\bullet,\bullet\bullet\rangle$ with $b=\alpha x+(\alpha+1)y+t$. By rotation, we get $a=(\alpha-1)y+z+\alpha x+t+y$. Hence, $x=t$ which is a contradiction.

So the coloring $\varphi\circ\mathcal{R}_{j+1}$ has configuration

$$\left\langle\bullet\,(\varphi\circ\mathcal{R}_{j+1})\,(1),(\varphi\circ\mathcal{R}_{j+1})\left(\frac{p}{2}+1\right)\circ\right\rangle$$

with a weighted sum of black vertices equal to $a=\alpha x+\alpha y+z+t$. This is the same situation as in the beginning.

This means that φ is $\frac{p}{2}$-antiperiodic. Hence the number $\alpha_x+\alpha_y+1$ of black vertices is $\frac{p}{2}$. Since $\frac{p}{2}=2\alpha+2$, $\frac{p}{2}$ must be even. But, as the cycle $\mathcal{C}_p$ is of Type 7, the number of vertices is such that $p=4q+2$ for some $q\in\mathbb{N}$. Thus, there is no possible constant 2-labelling in this case.

Case 3 : The configuration $\langle\bullet\bullet,\bullet\circ\rangle$ appears in φ. We have

$$a=\alpha_x x+\alpha_y y+z. \tag{2}$$

By $\frac{p}{2}$-rotation, we obtain the configuration $\langle\circ\bullet,\bullet\bullet\rangle$ with $b=\alpha_x y+\alpha_y x+t$ since $p\equiv 2 \pmod 4$. By rotation of this latter configuration, we have

$$a=(\alpha_x-1)x+t+(\alpha_y-1)y+z+y=(\alpha_x-1)x+\alpha_y y+z+t. \tag{3}$$

Since a is a constant, the equations (2) and (3) imply that $x=t$ which is impossible. So there is no posssible constant 2-labelling in this case.

Case 4 : The configuration $\langle\bullet\bullet,\circ\bullet\rangle$ appears in φ. As illustrated at Figure 7, this case is similar to Case 3 by axial symmetry where the axis is the diameter passing through the vertex 0, and by rotation $\mathcal{R}_{-1}$. Thus there is no possible constant 2-labelling in this case.

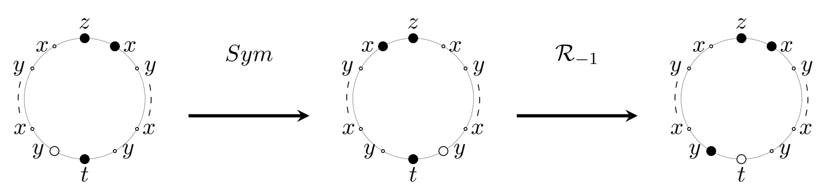


Figure 7: Axial symmetry Sym and (-1)-rotation $\mathcal{R}_{-1}$ of a coloring of Type 7 cycle $\mathcal{C}_p$.

□

Before considering the last type of cycles, we introduce a particular coloring φ of a cycle $\mathcal{C}_p$ with $p \equiv 0 \pmod 4$. We say that φ is *un joli petit nom* if

Lemma 11. *For cycles $\mathcal{C}_p$ of Type 8, i.e., $z(xy)^{\frac{p-4}{4}}xtx(yx)^{\frac{p-4}{4}}$ with $x \neq y \neq t$ and $4 < p \in \mathbb{N}$, if φ is a non trivial constant 2-labelling, then φ is one of the following,*

- *alternate with $a = (\frac{p}{2} - 2)y + z + t$ and $b = \frac{p}{2}x$,*
- *$\frac{p}{2}$-periodic with $a = (2\alpha+2)x+2\alpha y+z+t$ and $b = (2\alpha+2)(x+y)$ for $\alpha \in \{0, \dots, \frac{p}{4}-1\}$ such that the numbers of black vertices of weight x and y are equal when $\varphi(0) = \circ$,*
- *$\frac{p}{2}$-antiperiodic with $a = \frac{p}{4}x + (\frac{p}{4} - 1)y + z$ and $b = \frac{p}{4}x + (\frac{p}{4} - 1)y + t$,*
- *if $t = \frac{p}{4}x + (1 - \frac{p}{4})y$, then φ is, up to complementary, any rotation of a coloring ψ of $\mathcal{C}_p$ such that $\psi(i) = \psi(i + \frac{p}{2}) = \bullet$ for all even $i \in \{0, \dots, \frac{p}{2} - 1\}$ and $\psi(i) \neq \psi(i + \frac{p}{2})$ for all odd $i \in \{0, \dots, \frac{p}{2} - 1\}$. In this case, we have either $a = \frac{p}{2}x + (\frac{p}{4} - 1)y + z$ and $b = \frac{3p}{4}x$ or $a = (\frac{p}{4} - 1)y + z$ and $b = \frac{p}{4}x$.*

Note that the proof follows the same lines as the proof of Lemma 10.

Proof. By definition of Type 8, the number p of vertices is even and the alternate coloring is a constant 2-labelling of $\mathcal{C}_p$ with $a = (\frac{p}{2} - 2)y + z + t$ and $b = \frac{p}{2}x$. So let φ be a non trivial constant 2-labelling of $\mathcal{C}_p$ which is not alternate. We consider configurations of type $\langle \varphi(0)\varphi(1), \varphi(\frac{p}{2} + 1)\varphi(\frac{p}{2})\rangle$. Assume that $\varphi(0)\varphi(1) = \bullet\bullet$. We have four cases to consider.

Case 1 : The configuration $\langle \bullet\bullet, \bullet\bullet\rangle$ appears in φ. We have $a = \alpha_x x + \alpha_y y + z + t$ and by a rotation, we get $a = (\alpha_x - 2)y + z + t + \alpha_y x + x + x$ and so $\alpha_x = \alpha_y + 2$. Hence for $\alpha_x = \alpha_y := \alpha$, we have $a = (\alpha + 2)x + \alpha y + z + t$ with . $0 \leq \alpha < \frac{p-4}{2}$. Observe that the numbers of black vertices with respective weights x, y, z and t are constant for rotation of colorings with the configuration $\langle \bullet\bullet, \bullet\bullet\rangle$.

Let $i \in \{0, \dots, \frac{p}{2} - 1\}$ be the smallest integer such that $\varphi(i+1) = \circ$. As φ is non trivial, such an integer exists. We can assume that

$$\varphi\left(\frac{p}{2} + \ell\right) = \bullet \qquad \forall \ell \in \{0, \dots, i\}.$$

Otherwise, we consider the coloring $\varphi \circ \mathcal{R}_{\frac{p}{2}}$ instead of φ.

It implies that $\varphi\left(\frac{p}{2} + i + 1\right) = \circ$. Indeed, assume that $\varphi\left(\frac{p}{2} + i + 1\right) = \bullet$ as illustrated at Figure 8. Then the coloring $\varphi \circ \mathcal{R}_i$ has configuration $\langle \bullet\circ, \bullet\bullet\rangle$ with $a = (\alpha+2)x + \alpha y + z + t$. Since a $\frac{p}{2}$-rotation maps vertices of weight x (respectively y) on vertices of weight x (resp. y), the coloring $\varphi \circ \mathcal{R}_{i+\frac{p}{2}}$ has configuration $\langle \bullet\bullet, \circ\bullet\rangle$ with $a = (\alpha + 2)x + \alpha y + z + t$. By rotation, we obtain $a = (\alpha+1)y + z + \alpha x + x + x = (\alpha+2)x + (\alpha+1)y + z$. It implies $y = t$ which is a contradiction.

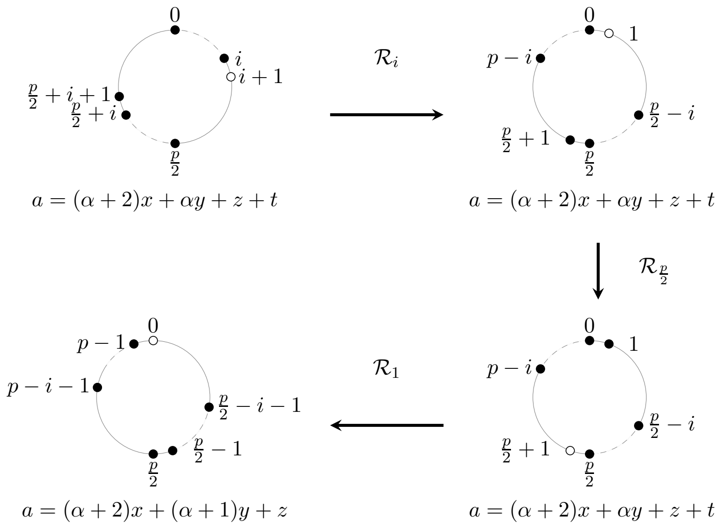


Figure 8: Rotations of a coloring of Type 8 cycle $\mathcal{C}_p$ and their corresponding weighted sums of black vertices.

Hence $\varphi(i+1)=\varphi\left(\frac{p}{2}+i+1\right)=\circ$ and the coloring $\varphi\circ\mathcal{R}_{i+1}$ has a weighted sum of black vertices equal to $b=(\alpha+2)(x+y)$. Observe that for a coloring with configuration $\langle\circ\circ,\circ\circ\rangle$ and weighted sum $b=(\alpha+2)(x+y)$, we obtain again $b=(\alpha+2)(y+x)$ after rotation.

Let $j\in\{i+1,\dots,\frac{p}{2}-1\}$ be the smallest integer such that $\varphi(j+1)=\bullet$. Without loss of generality, we can assume that

$$\varphi\left(\frac{p}{2}+\ell\right)=\circ \qquad \forall\ell\in\{i+1,\dots,j\}.$$

Therefore $\varphi\left(\frac{p}{2}+j+1\right)=\bullet$. Indeed, assume that $\varphi\left(\frac{p}{2}+j+1\right)=\circ$. Then the coloring $\varphi\circ\mathcal{R}_j$ has configuration $\langle\circ\bullet,\circ\circ\rangle$ with $b=(\alpha+2)(x+y)$. By rotation, we get $a=(\alpha+1)y+z+(\alpha+2)x$. Hence, $y=t$ which is a contradiction.

So the coloring $\varphi\circ\mathcal{R}_{j+1}$ has configuration

$$\left\langle\bullet\left(\varphi\circ\mathcal{R}_{j+1}\right)(1),\left(\varphi\circ\mathcal{R}_{j+1}\right)\left(\frac{p}{2}+1\right)\bullet\right\rangle$$

with $a=(\alpha+2)x+\alpha y+z+t$. This is the same situation as in the beginning.

Thus φ is $\frac{p}{2}$-periodic. We have $a=(\alpha+2)x+\alpha y+z+t$ and $b=(\alpha+2)(x+y)$ for $\alpha\in\{0,\dots,\frac{p}{2}-3\}$.

Case 2 : The configuration $\langle\bullet\bullet,\circ\circ\rangle$ appears in φ. We have $a=\alpha_x x+\alpha_y+z$ and by a rotation, we get $a=(\alpha_x-1)y+z+\alpha_y x+x$ and $\alpha_x=\alpha_y+1$. Set $\alpha:=\alpha_y$. We have $a=(\alpha+1)x+\alpha y+z$ with $1\le\alpha<\frac{p-2}{2}$. Moreover we know the value of the constant b as the coloring $\varphi\circ\mathcal{R}_{\frac{p}{2}}$ has configuration $\langle\circ\circ,\bullet\bullet\rangle$ with a weighted sum of black vertices equal

to $b = (\alpha+1)x + \alpha y + t$. Observe that the numbers of black vertices with respective weights x, y, z and t are constant for rotation of colorings with the configuration $\langle \bullet\bullet, \circ\circ \rangle$.

Let $i \in \{0, \ldots, \frac{p}{2} - 1\}$ be the smallest integer such that $\varphi(i+1) = \circ$. We can assume that

$$\varphi\left(\frac{p}{2} + \ell\right) = \circ \qquad \forall \ell \in \{0, \ldots, i\}.$$

Otherwise, we consider the coloring $\varphi \circ \mathcal{R}_{\frac{p}{2}}$ instead of φ and we apply the same reasoning with a swapping of the colors.

Assume that $\varphi\left(\frac{p}{2} + i + 1\right) = \circ$. Then the coloring $\varphi \circ \mathcal{R}_i$ has configuration $\langle \bullet\circ, \circ\circ \rangle$ with $a = (\alpha+1)x + \alpha y + z$. By rotation, we obtain $b = (\alpha+1)y + \alpha x + x$. It implies $y = t$ which is a contradiction. Hence $\varphi\left(\frac{p}{2} + i + 1\right) = \bullet$. Observe that for any coloring with configuration $\langle \circ\circ, \bullet\bullet \rangle$ and weighted sum $b = (\alpha+1)x + \alpha y + t$, we obtain again $b = \alpha y + t + \alpha x + x = (\alpha+1)x + \alpha y + t$ after rotation.

Let $j \in \{i+1, \ldots, \frac{p}{2} i\}$ be the smallest integer such that $\varphi(j+1) = \bullet$. Without loss of generality, we can assume that

$$\varphi\left(\frac{p}{2} + \ell\right) = \bullet \qquad \forall \ell \in \{i+1, \ldots, j\}.$$

Therefore $\varphi\left(\frac{p}{2} + j + 1\right) = \bullet$. Indeed, assume that $\varphi\left(\frac{p}{2} + j + 1\right) = \circ$. Then the coloring $\varphi \circ \mathcal{R}_j$ has configuration $\langle \circ\bullet, \bullet\bullet \rangle$ with $b = (\alpha+1)x + \alpha y + t$. By rotation, we get $a = (\alpha-1)y + z + t + \alpha x + x = (\alpha+1)x + (\alpha-1)y + z + t$. Hence, $y = t$ which is a contradiction.

So the coloring $\varphi \circ \mathcal{R}_{j+1}$ has configuration

$$\left\langle \bullet \left(\varphi \circ \mathcal{R}_{j+1}\right)(1), \left(\varphi \circ \mathcal{R}_{j+1}\right)\left(\frac{p}{2} + 1\right) \circ \right\rangle$$

with a weighted sum of black vertices equal to $a = (\alpha+1)x + \alpha y + z$. This is the same situation as in the beginning.

Hence, φ is $\frac{p}{2}$-antiperiodic. The number of black vertices in the coloring is $\frac{p}{2} 2\alpha + 2$. Note that as $\mathcal{C}_p$ is of type, $p \equiv 0 \pmod 4$ and $\frac{p}{2}$ is even. So, we have $a = \frac{p}{4}x + (\frac{p}{4} - 1)y + z$ and $b = \frac{p}{4}x + (\frac{p}{4} - 1)y + t$.

<u>Case 3</u> : The configuration $\langle \bullet\bullet, \bullet\circ \rangle$ appears in φ. We have $a = \alpha_x x + \alpha_y y + z$ and by rotation, we get $a = (\alpha_x - 2)y + z + t + \alpha_y x + x$. Hence,

$$t = (\alpha_x - \alpha_y - 1)x + (\alpha_y - \alpha_x + 2)y. \tag{4}$$

Moreover, we know the value of the constant b by a $\frac{p}{2}$-rotation. The coloring $\varphi \circ \mathcal{R}_{\frac{p}{2}}$ has weighted sum of black vertices equal to $b = \alpha_x x + \alpha_y y + t$.

Consider the coloring $\varphi \circ \mathcal{R}_1$. It has weighted sum $a = (\alpha_y + 1)x + (\alpha_x - 2)y + z + t$ and two possible configurations : either $\langle \bullet\circ, \bullet\bullet \rangle$ or $\langle \bullet\bullet, \circ\bullet \rangle$. Indeed, the configurations $\langle \bullet\bullet, \bullet\bullet \rangle$ and $\langle \bullet\circ, \circ\bullet \rangle$ are forbidden as they imply by rotation respectively $a = (\alpha_y - 1)y + z + t + (\alpha_x - 2)x + 2x$ and $b = (\alpha_y + 1)y + (\alpha_x - 2)x + 2x$. Hence they both imply $y = t$ which is a contradiction.

If $\varphi \circ \mathcal{R}_1$ has configuration $\langle \bullet\circ, \bullet\bullet \rangle$, then the coloring $\varphi \circ \mathcal{R}_2$ has weighted sum $b = \alpha_x x + \alpha_y y + t$. The only possible configuration for $\varphi \circ \mathcal{R}_2$ is $\langle \circ\bullet, \bullet\bullet \rangle$. Indeed, if the configuration of $\varphi \circ \mathcal{R}_2$ is in $\{\langle \circ\circ, \bullet\bullet \rangle, \langle \circ\circ, \circ\bullet \rangle, \langle \circ\bullet, \circ\bullet \rangle\}$, then it implies by rotation that $y = t$, which is a contradiction. Hence, in this case, we obtain that the weighted sum of $\varphi \circ \mathcal{R}_3$ is $a = (\alpha_x - 2)y + z + t + \alpha_y x + x$.

If $\varphi \circ \mathcal{R}_1$ has configuration $\langle \bullet\bullet, \circ\bullet \rangle$, then $\varphi \circ \mathcal{R}_2$ has weighted sum $a = \alpha_x x + \alpha_y y + z$. Using the same reasoning as above, the only possible configuration for $\varphi \circ \mathcal{R}_2$ is $\langle \bullet\bullet, \bullet\circ \rangle$ and the weighted sum of $\varphi \circ \mathcal{R}_3$ is $a = (\alpha_x - 2)y + z + t + \alpha_y x + x$.

Therefore, for both cases of configurations of $\varphi \circ \mathcal{R}_1$, we have that $\varphi \circ \mathcal{R}_3$ has configuration

$$\left\langle \bullet\, \varphi \circ \mathcal{R}_3\,(1)\,, \varphi \circ \mathcal{R}_3 \left(\frac{p}{2} + 1\right) \bullet \right\rangle$$

and weighted sum of black vertices $a = (\alpha_y + 1)x + (\alpha_x - 2)y + z + t$. Hence, we get the same situation as for $\varphi \circ \mathcal{R}_1$ and we can apply the same reasoning again.

This means that a black pair of diametrically opposed vertices is always followed by a white and black pair and vice versa. Note that this is possible since $p \equiv 0 \pmod 4$. In this case, we have $\alpha_x = \frac{p}{2}$, $\alpha_y = \frac{p}{4} - 1$ and, by (4),

$$t = \left(\frac{p}{2} - \frac{p}{4}\right) x + \left(\frac{p}{4} - \frac{p}{2} + 1\right) y = \frac{p}{4} x + \left(1 - \frac{p}{4}\right) y.$$

So we obtain $a = \frac{p}{2}x + (\frac{p}{4} - 1)y + z$ and $b = \frac{3p}{4}x$. By Proposition 2, we get $a = (\frac{p}{4} - 1)y + z$ and $b = \frac{p}{4}x$ for the complementary coloring since

$$\sum_{u \in \{0,\ldots,p-1\}} w(u) = z + t + \frac{p}{2}x + \left(\frac{p}{2} - 2\right) y.$$

Case 4 : The configuration $\langle \bullet\bullet, \circ\bullet \rangle$ appears in φ. This case is similar to Case 3 by axial symmetry where the axis is the diameter passing through the vertex 0 and by rotation $\mathcal{R}_{-1}$. Hence we obtain the same conclusion. $\square$

Using all the previous lemmas, we can now prove our main theorem.

Theorem 12. *Let φ be a non trivial constant 2-labelling of a cycle $\mathcal{C}_p$ of Type 1–8 with $A = \{\mathcal{R}_k \mid k \in \mathbb{Z}\}$ and $v = 0$. Let $a = \sum_{\{u \in V \mid \varphi \circ \xi(u) = \bullet\}} w(u)$ and $b = \sum_{\{u \in V \mid \varphi \circ \xi'(u) = \bullet\}} w(u)$ for $\xi \in A_\bullet, \xi' \in A_\circ$. We have the following possible values of the constants a and b depending of the type of $\mathcal{C}_p$:*

Type	*Value of a*	*Value of b*	*Condition on parameters*
1	$\alpha x + z$	$(\alpha+1)x$	$\alpha \in \{0,\dots,p-2\}$
2	$2\alpha x + t + z$	$2(\alpha+1)x$	$\alpha \in \{0,\dots,\frac{p-4}{2}\}$
	$(\frac{p}{2}-1)x + z$	$(\frac{p}{2}-1)x + t$	
3	$\emptyset$	$\emptyset$	
4	$(\alpha+1)x + \alpha y + z$	$(\alpha+1)(x+y)$	$\alpha \in \{0,\dots,\frac{p-4}{2}\}$
	$(\frac{p}{2}-1)y + z$	$\frac{p}{2}x$	
5	$\frac{p}{3}x + (\frac{p}{3}-1)y + z$	$(\frac{p}{3}-1)x + (\frac{p}{3}+1)y$	$p \equiv 0 \pmod 3$
6	$\frac{p}{3}x + (\frac{p}{3}-1)y + z$	$(\frac{p}{3}+1)x + (\frac{p}{3}-1)y$	$p \equiv 0 \pmod 3$
7	$a = (\frac{p}{2}-1)y + z$	$b = (\frac{p}{2}-1)x + t$	
	$a = \alpha(x+y) + t + z$	$b = (\alpha+1)(x+y)$	$\alpha \in \{0,\dots,\frac{p}{2}-1\}$
8	$a = (\frac{p}{2}-2)y + z + t$	$b = \frac{p}{2}x$	
	$a = (2\alpha+2)x + 2\alpha y + z + t$	$b = (2\alpha+2)(x+y)$	$\alpha \in \{0,\dots,\frac{p}{4}-1\}$
	$a = \frac{p}{4}x + (\frac{p}{4}-1)y + z$	$b = \frac{p}{4}x + (\frac{p}{4}-1)y + t$	
	$a = \frac{p}{2}x + (\frac{p}{4}-1)y + z$	$b = \frac{3p}{4}x$	$t = \frac{p}{4}x + (1-\frac{p}{4})y$
	$a = (\frac{p}{4}-1)y + z$	$b = \frac{p}{4}x$	$t = \frac{p}{4}x + (1-\frac{p}{4})y$

2 Codes

In this section, we consider the graph of the infinite grid $\mathbb{Z}^2$. The vertices are all pairs of integers and two vertices (x_1, x_2) and (y_1, y_2) are adjacent if $|x_1 - y_1| + |x_2 - y_2| = 1$. The infinite grid is a 4-regular graph, *i.e.*, every vertex has 4 neighbours. Let the sets $L_e = \{(x_1, x_2) \in \mathbb{Z}^2 \mid x_1 + x_2 = 0 \pmod 2\}$ and $L_o = \{(x_1, x_2) \in \mathbb{Z}^2 \mid x_1 + x_2 = 1 \pmod 2\}$ denote the *even* and *odd* sub-lattices of $\mathbb{Z}^2$. Sets $\{(x_1, x_1 + c) \mid x_1 \in \mathbb{Z}\}$ and $\{(x_1, -x_1 + c) \mid x_1 \in \mathbb{Z}\}$ with $c \in \mathbb{Z}$ are called *diagonals* of $\mathbb{Z}^2$.

Recall that for a graph $G = (V, E)$ and a positive integer r, a set $S \subseteq V$ of vertices is an (r, a, b)-*code* if every element of S belongs to exactly a balls of radius r centered at elements of S and every element of $V \setminus S$ belongs to exactly b balls of radius r centered at elements of S. For the infinite grid $\mathbb{Z}^2$, we consider balls defined relative to the Manhattan metric. The distance between two points $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ of $\mathbb{Z}^2$ is $d(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + |x_2 - y_2|$. We can view an (r, a, b)-code of $\mathbb{Z}^2$ as a particular coloring φ with two colors black and white where the black vertices are the elements of the code. In other words, the coloring φ is such that a ball of radius r centered on a black (resp. white) vertex contain exactly a (resp. b) black vertices.

Firstly, we present the *projection* and *folding* method used to prove our Theorem 16.

Note that to apply this method, the coloring of the grid must satisfy some specific properties. Secondly, we show how the method can be used in the case of (r, a, b)-codes of $\mathbb{Z}^2$. Finally, we give the proof of Theorem 16 using Axenovich's results.

2.1 Projection and folding

Let $r, t \in \mathbb{Z}$ and $\varphi : \mathbb{Z}^2 \to \{\circ, \bullet\}$ be a coloring of $\mathbb{Z}^2$ such that the coloring of a line is obtained by doing a translation $\mathbf{t} = (t, 1)$ (resp. -$\mathbf{t} = (-t, -1)$) of the coloring of the line below (resp. above). In this case, if we know the coloring of one line and the translation $\mathbf{t}$, then the coloring of $\mathbb{Z}^2$ is known. In particular, for any vertex $\mathbf{x} \in \mathbb{Z}^2$, we have $\varphi(\mathbf{x}) = \varphi(\mathbf{x} + \mathbf{t})$. Assume moreover that φ is such that $\varphi(\mathbf{x}) = \varphi(\mathbf{x} + (p, 0))$ for some $p \in \mathbb{Z}$ and all $\mathbf{x} \in \mathbb{Z}^2$. Suppose that p is the smallest integer satisfying this property.

Projection

Let $\mathbf{y} \in \mathbb{Z}^2$. Using the translation $\mathbf{t} = (t, 1)$, we can project the ball $B_r(\mathbf{y})$ on the line L containing $\mathbf{y}$. For easier notation, assume $\mathbf{y} = (0, 0)$. Let $Trans$ denote the set of all translated of $B_r(\mathbf{y})$ by a multiple of $\mathbf{t}$. Let $h : L \to \mathbb{N}$ be a map defined by

$$h((i, 0)) = \#\{T \in Trans \mid (i, 0) \in T\}.$$

The image of the line L by the mapping h, denoted by $h(L)$, is the *projection* of $B_r(\mathbf{y})$ with translation $\mathbf{t} = (t, 1)$. Note that $h((i, 0)) < \infty$ and h has a non zero value only finitely many times (see Figure 10 for example). This map is introduced to count the number of occurences in the ball $B_r(\mathbf{y})$ of vertices of L, up to translation $\mathbf{t}$. Observe that $\sum_{i \in \mathbb{Z}} h((i, 0)) = 2r^2 + 2r + 1$ since a ball of radius r contains exactly $2r^2 + 2r + 1$ vertices.

Folding

Using the translation $(p, 0)$, *i.e.*, $\varphi(\mathbf{x}) = \varphi(\mathbf{x} + (p, 0))$ for all $\mathbf{x} \in \mathbb{Z}^2$, we can fold a projection on a cycle of p weighted vertices. Let L be the line containing $\mathbf{y} = (0, 0)$ and $\{0, \ldots, p-1\}$ be the set of vertices of the cycle $\mathcal{C}_p$. We define a map $w : \{0, \ldots, p-1\} \to \mathbb{N}$ such that, for $i \in \{0, \ldots, p-1\}$,

$$w(i) := \sum_{k \in \mathbb{Z}} h((i + kp, 0)).$$

The *folding* of the projection $h(L)$ is the cycle $\mathcal{C}_p$ with vertices $0, \ldots, p-1$ of respective weights $w(0), \ldots, w(p_1)$.

Example 1. Consider balls of radius 3 of the infinite grid (see Figure 9). Assume that the coloring φ of $\mathbb{Z}^2$ satisfies the translations $\mathbf{t} = (2, 1)$ and $(p, 0) = (5, 0)$, *i.e.*,

$$\varphi(\mathbf{x}) = \varphi(\mathbf{x} + (2, 1)) \text{ and } \varphi(\mathbf{x}) = \varphi(\mathbf{x} + (5, 0)) \qquad \forall \mathbf{x} \in \mathbb{Z}^2.$$

For $\mathbf{y} \in \mathbb{Z}^2$, we can compute the projection of $B_r(\mathbf{y})$ with translation $(2, 1)$ (see Figure 10) and its folding on a cycle $\mathcal{C}_5$ (see Figure 11).

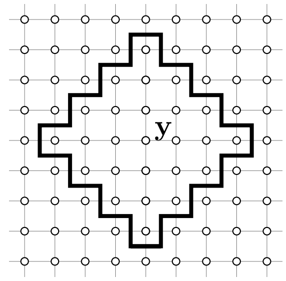


Figure 9: In $\mathbb{Z}^2$, a ball $B_3(\mathbf{x})$ centered on $\mathbf{x}$ and of radius 3.

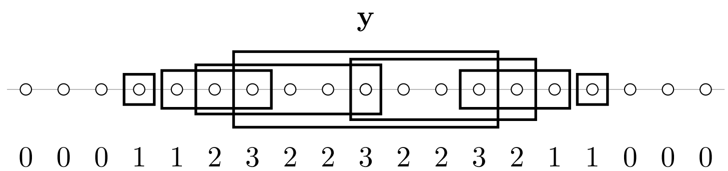


Figure 10: Projection on a line of $\mathbb{Z}^2$ of a ball $B_3(\mathbf{y})$ centered on $\mathbf{y}$ and of radius 3 with the translation $\mathbf{t} = (2, 1)$. The rectangles indicate the intersection of the line and elements of the set $Trans$. Under the line is the image of the line by the mapping h.

2.2 Application to (r, a, b)-codes

In order to use our projection and folding method for (r, a, b)-codes, we recall the notions of periodic colorings and diagonal colorings in the graph of the infinite grid $\mathbb{Z}^2$. Let $\mathbf{u}, \mathbf{v}$ be two vectors. A coloring φ of $\mathbb{Z}^2$ is *periodic* if $\mathbf{u}$ and $\mathbf{v}$ are non-colinear and $\varphi(\mathbf{x} + \mathbf{u}) = \varphi(\mathbf{x}) = \varphi(\mathbf{x} + \mathbf{v})$ for all $\mathbf{x} \in \mathbb{Z}^2$.

Theorem 13 (Puzynina [10])**.** *For $r \geq 2$, every (r, a, b)-code of $\mathbb{Z}^2$ is periodic.*

In particular, if φ is a periodic coloring, then we have, for some $m, n \in \mathbb{Z}$, $\varphi(\mathbf{x}+(m, 0)) = \varphi(\mathbf{x}) = \varphi(\mathbf{x} + (0, n))$ for all $\mathbf{x} \in \mathbb{Z}^2$.

Proposition 14 (Axenovich [2])**.** *If φ is an (r, a, b)-code of $\mathbb{Z}^2$ with $r \geq 2$ and $|a - b| > 4$, then it is a diagonal coloring, i.e., φ is such that the even and odd sublattices are the disjoint unions of monochromatic diagonals.*

Let $r \geq 2$ and $a, b \in \mathbb{N}$ such that $|a - b| > 4$. Let φ be an (r, a, b)-code of $\mathbb{Z}^2$. By Proposition 14, φ is a diagonal coloring . Hence, φ is determined by the coloring of any horizontal line, e.g. $\{(x_1, 0) \mid x_1 \in \mathbb{Z}\}$, and by the orientation of the monochromatic diagonals in the even and odd sublattices (see Figure 12).

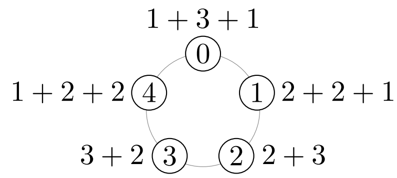


Figure 11: The folding of a ball $B_3(\mathbf{y})$ with translation $\mathbf{t} = (2, 1)$ on the cycle $\mathcal{C}_5$.

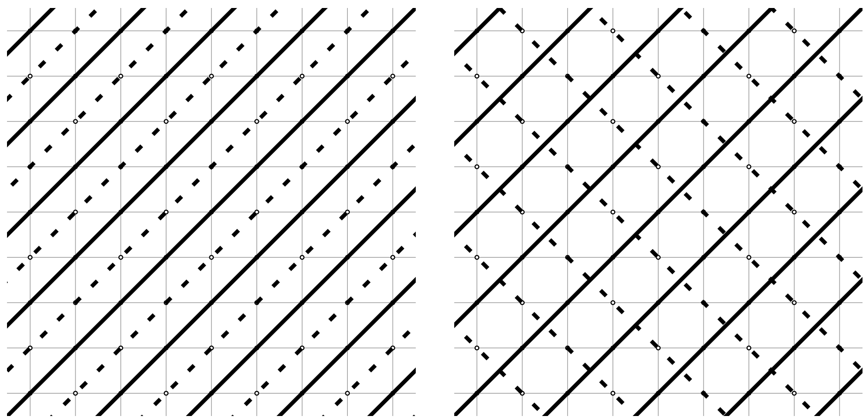


Figure 12: Schemes of diagonal colorings of the infinite grid with respectively parallel and non parallel monochromatic diagonals.

Assume first that the monochromatic diagonals are all parallel. Without loss of generality, we can suppose that they are of the type $\{(x_1, x_1 + c) \mid x_1 \in \mathbb{Z}\}$ with $c \in \mathbb{Z}$. Indeed, the case where the monochromatic diagonals are of type $\{(x_1, -x_1 + c) \mid x_1 \in \mathbb{Z}\}$ is similar since the grid is symmetric. In this case, if the coloring of a line of $\mathbb{Z}^2$ is known, then the coloring of the line above (resp. below) is obtained by doing a translation $\mathbf{t} = (1, 1)$ (resp. $-\mathbf{t}$) as $\varphi(\mathbf{x}) = \varphi(\mathbf{x} + \mathbf{t})$ for all $\mathbf{x} \in \mathbb{Z}^2$. So we can apply the projection method. Moreover, by Theorem 13, φ is such that $\varphi(\mathbf{x} + (m, 0)) = \varphi(\mathbf{x})$ for some $m \in \mathbb{N}$ and all $\mathbf{x} \in \mathbb{Z}^2$. Hence, it is possible to apply the folding method.

Now assume that the monochromatic diagonals are not parallel. We may suppose that the even (resp. odd) sublattice is the union of monochromatic diagonals of type $\{(x_1, x_1 + c) \mid x_1 \in \mathbb{Z}\}$ (resp. $\{(x_1, -x_1 + c) \mid x_1 \in \mathbb{Z}\}$) with $c \in \mathbb{Z}$. We consider a ball $B_r(\mathbf{x})$ centered on $\mathbf{x}$ and of radius r. Observe that a diagonal intersecting the ball contains either r or $r + 1$ elements. Moreover two intersecting diagonals belong to the same sublattice. Hence, in terms of counting vertices of a particular color appearing in the ball, it is equivalent to consider monochromatic diagonals that are parallel or not. So, we can apply the folding

method in both cases.

Therefore, for $r \geq 2$ and $|a-b| > 4$, there exists an (r,a,b)-code of the infinite grid $\mathbb{Z}^2$ if and only if there exists a constant 2-labelling of some cycle $\mathcal{C}_p$, with $v = 0$, $A = \{\mathcal{R}_k \mid k \in \mathbb{Z}\}$ and a mapping w defined as before, such that

$$a = \sum_{\{u \in V \mid \varphi \circ \xi(u) = \bullet\}} w(u) \text{ and } b = \sum_{\{u \in V \mid \varphi \circ \xi' = \bullet\}} w(u) \quad \forall \xi \in A_\bullet,\ \xi' \in A_\circ.$$

2.3 Characterization of (r,a,b)-codes of $\mathbf{Z}^2$ with $|a-b| > 4$ and $r \geq 2$

For $r \geq 2$ and $|a-b| > 4$, Axenovich described all possible (r,a,b)-codes (see [2]). Note that a diagonal coloring φ of $\mathbb{Z}^2$ is called *q-periodic* (resp. *q-antiperiodic*) if horizontal lines are colored q-periodically (resp. q-antiperiodically), *i.e.*,

$$\varphi((x_1,x_2)) = \varphi(x_1+q,x_2)) \text{ (resp. } \varphi((x_1,x_2)) \neq \varphi(x_1+q,x_2))) \qquad \forall (x_1,x_2) \in \mathbb{Z}^2.$$

Theorem 15 (Axenovich [2])**.** *If a coloring is an (r,a,b)-code with $r \geq 2$ and $|a-b| > 4$, then it is one of the following diagonal Colorings 1–5 :*

1. *q-periodic coloring where $q \in \{r, r+1\}$ is odd and the monochromatics diagonal are parallel.*
2. *q-antiperiodic coloring where $q \in \{r, r+1\}$ is even.*
3. *q-periodic coloring where $q \in \{r, r+1\}$ is even and for all horizontal or vertical interval I of length p the number of black vertices from the even sublattice and from the odd sublattice is the same.*
4. *$(2r+1)$-periodic coloring and for all horizontal or vertical interval I of length p the number of black vertices from the even sublattice and from the odd sublattice is the same.*
5. *2-periodic or 3-periodic coloring.*

This theorem allows us to apply the projection and folding method in this case. Let $\mathbf{y} = (0,0)$. By Theorem 15, we fold the ball $B_r(\mathbf{y})$ with translation $\mathbf{t} = (1,1)$ on cycles $\mathcal{C}_p$ with $p \in \{2, 3, r, r+1, 2r, 2r+1, 2r+2\}$ accordingly with Colorings 1–5. So we consider the projection of $B_r(\mathbf{y})$ on the line L with translation $\mathbf{t} = (1,1)$. We obtain for an even (resp. odd) radius r,

$$h((i,0)) = \begin{cases} r & \text{if } i \leq r \text{ and } i \text{ is odd (resp. even)} \\ r+1 & \text{if } i \leq r \text{ and } i \text{ is even (resp. odd)} \\ 0 & \text{otherwise} \end{cases}$$

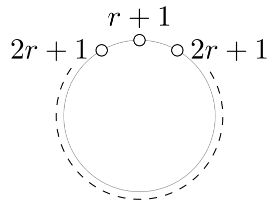
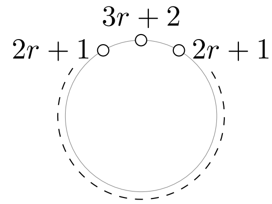
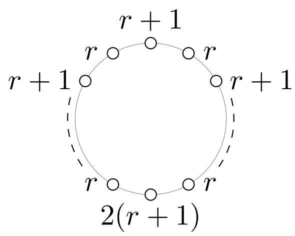
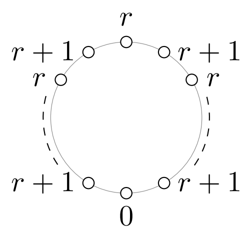
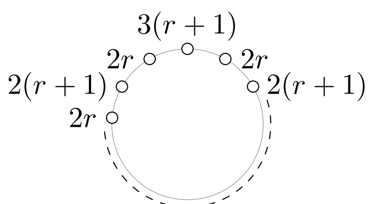
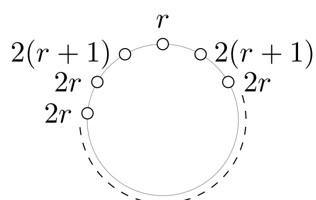
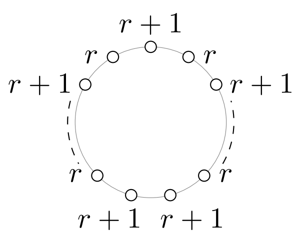
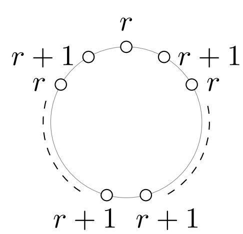
	For r even	For r odd
Coloring 1	Type 1 : $p = r + 1$ 	Type 1 : $p = r$ 
Coloring 2	Type 8 : $p = 2r$ 	Type 8 : $p = 2(r + 1)$ 
Coloring 3	Type 4 : $p = r$ 	Type 4 : $p = r + 1$ 
Coloring 4	Type 5 : $p = 2r + 1$ 	Type 6 : $p = 2r + 1$ 

Figure 13: Weighted cycles $\mathcal{C}_p$ corresponding to Colorings 1–4.

Indeed, if r is even, then any diagonal of the even (resp. odd) sublattice intersecting the ball contains $r+1$ (resp. r) elements of $B_r(\mathbf{y})$. The other case can be treated similarly.

Consider now Colorings 1–5. For each kind of coloring, we determine the projection and folding of $B_r(\mathbf{y})$ on the cycle $\mathcal{C}_p$ according to the parity of r (see Figure 13).

The coloring 1 gives two different weighted cycles. If r is even, then $B_r(\mathbf{y})$ is projected and folded on the cycle $\mathcal{C}_{r+1}$ of Type 1 with $z = r+1$ and $x = 2r+1$. If r is odd,$B_r(\mathbf{y})$ is projected and folded on the cycle $\mathcal{C}_r$ of Type 1 with $z = 3r+2$ and $x = 2r+1$.

The coloring 2 gives two different weighted cycles. If r is even, $B_r(\mathbf{y})$ is projected and folded on the cycle $\mathcal{C}_{2r}$ of Type 8 with $z = r+1 = y$, $x = r$ and $t = 2r+2$. If r is odd, $B_r(\mathbf{y})$ is projected and folded on the cycle $\mathcal{C}_{2r+2}$ of Type 8 with $z = r = y$, $x = r+1$ and $t = 0$.

The coloring 3 gives two different weighted cycles. If r is even, $B_r(\mathbf{y})$ is projected and folded on the cycle $\mathcal{C}_r$ of Type 4 with $z = 3(r+1), x = 2r$ and $y = 2(r+1)$. If r is odd, $B_r(\mathbf{y})$ is projected and folded on the cycle $\mathcal{C}_{r+1}$ of Type 4 with $z = r, x = 2(r+1)$ and $y = 2r$.

The coloring 4 gives two different weighted cycles. If r is even, $B_r(\mathbf{y})$ is projected and folded on the cycle $\mathcal{C}_{2r+1}$ of Type 5 with $z = r+1 = y$ and $x = r$. If r is odd, $B_r(\mathbf{y})$ is projected and folded on the cycle $\mathcal{C}_{2r+1}$ of Type 6 with $z = r = y$ and $x = r+1$.

The coloring 5 gives five different weighted cycles. If φ is 2-periodic, then $B_r(\mathbf{y})$ is projected and folded on $\mathcal{C}_2$ of Type 1 with $z = (r+1)^2$ and $x = r^2$ for r even and with $z = r^2$ and $x = (r+1)^2$ for r odd. If φ is 3-periodic, then $B_r(\mathbf{y})$ is projected and folded on $\mathcal{C}_3$ of Type 1. In that case, straightforward analysis give the weights z and x :

- $z = \frac{2r^2+2r-1}{3}$ and $x = \frac{2r^2+2r+2}{3}$ if $r = 3k+1$,
- $z = \frac{2r^2+2r}{3} - 2k + 1$ and $x = \frac{2r^2+2r}{3} + k$ if $r = 3k-1$,
- $z = \frac{2r^2+2r}{3} + 2k + 1$ and $x = \frac{2r^2+2r}{3} - k$ if $r = 3k$.

Theorem 12 gives the possible values of the constants if

$$a = \sum_{\{u \in V | \varphi \circ \xi(u) = \bullet\}} w(u) \text{ and } b = \sum_{\{u \in V | \varphi \circ \xi' = \bullet\}} w(u) \quad \forall \xi \in A_\bullet,\ \xi' \in A_\circ.$$

This concludes the proof of Theorem 16.

Theorem 16. *Let $r, a, b \in \mathbb{N}$ such that $|a-b| > 4$ and $r \geq 2$. For all (r,a,b)-codes of $\mathbb{Z}^2$, the values of a and b are given in the following table.*

	a	b	*Condition on parameters*
Coloring 1			
r *even*	$r+1+\alpha(2r+1)$	$(\alpha+1)(2r+1)$	$\alpha \in \{0,\dots,r-1\}$
r *odd*	$3r+2+\alpha(2r+1)$	$(\alpha+1)(2r+1)$	$\alpha \in \{0,\dots,r-2\}$
Coloring 2			
r *even*	$\frac{r}{2}(2r+1)$	$\frac{r^2}{2}+(\frac{r}{2}+1)(r+1)$	
r *odd*	$\frac{r+1}{2}(2r+1)$	$\frac{(r+1)^2}{2}+(\frac{r+1}{2}-1)r$	
Coloring 3			
r *even*	$2(\alpha+1)r+(2\alpha+3)(r+1)$	$2(\alpha+1)(2r+1)$	$\alpha \in \{0,\dots,\frac{r-4}{2}\}$
r *even*	$(r+1)^2$	r^2	
r *odd*	$2(\alpha+1)(r+1)+(2\alpha+1)r$	$2(\alpha+1)(2r+1)$	$\alpha \in \{0,\dots,\frac{r-3}{2}\}$
r *odd*	r^2	$(r+1)^2$	
Coloring 4			
	$\frac{(2r+1)^2}{3}$	$\frac{(2r+1)^2}{3}+1$	$2r+1 \equiv 0 \pmod 3$
Coloring 5			
r *even*	$(r+1)^2$	r^2	
r *odd*	r^2	$(r+1)^2$	
$r=3k+1$	$\frac{2r^2+2r-1}{3}+\alpha\frac{2r^2+2r+2}{3}$	$(\alpha+1)\frac{2r^2+2r+2}{3}$	$\alpha \in \{0,1\}$
$r=3k-1$	$\frac{2r^2+2r}{3}-2k+1+\alpha\frac{2r^2+2r}{3}+k$	$(\alpha+1)\frac{2r^2+2r}{3}+k$	$\alpha \in \{0,1\}$
$r=3k$	$\frac{2r^2+2r}{3}+2k-1+\alpha\frac{2r^2+2r}{3}-k$	$(\alpha+1)\frac{2r^2+2r}{3}-k$	$\alpha \in \{0,1\}$

3 Conclusions and perspectives

Observe that for Theorems 12 and 16, we obtain not only numerical values for the constants but also the descriptions of the possible colorings. Moreover, the projection and folding method is presented in general and can be applied to linear codes. It would be interesting to consider (r, a, b)-codes in other types of lattices, for example, in the king lattice.

In [6, Theorem 4], Dorbec *et al.* present a method to construct $(1, a, b)$-codes in $\mathbb{Z}^d$. This method is based on a one-dimensional pattern of finite length that is extended by translations to color $\mathbb{Z}^d$. If there exists a similar method for (r, a, b)-codes with $r \geq 2$, then we could apply the projection and folding method in higher dimensions. In $\mathbb{Z}^d$, we conjecture that for colorings satisfying certain periodicity properties the projection and folding method will lead to weighted cycles with more distinct weights. A good strategy to handle this problem could be to find all possible weighted cycles that yield to constant 2-labellings where A is the set of rotations. It will give a generalization of Theorem 12 for all weighted cycles.

The problem of finding a constant 2-labelling of a graph is interesting in itself. On one hand, we could study constant 2-labelling in graphs having a big automorphisms group, for instance, in circulant graphs or in vertex-transitive graphs. On the other hand, we could find a natural generalization of constant 2-labellings into constant k-labellings using k colors and then consider their links with distinguishing numbers and weighted codes with more

than two values.

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